Problem 16.11

[Computer] Make plots similar to Figure 16.8 of the wave of Example 16.2 but from $t = 0$ to τ, the period, and for more closely spaced times. Animate your pictures and describe the motion.

Solution

In Problem 16.9 the general solution to the initial boundary value problem,

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ -\infty < t < \infty \\ &u(0,t) = 0 \\ &u(L,t) = 0 \\ &u(x,0) = f(x) \\ &\frac{\partial u}{\partial t}(x,0) = g(x), \end{aligned}$$

was found to be

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

where

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \\ B_n &= \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx. \end{aligned}$$

For a string released from rest initially with the shape of the triangular wave illustrated in Figure 16.7,

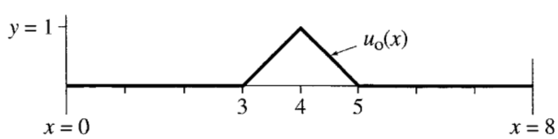


Figure 16.7 A string is released from rest at $t = 0$ in the triangular position shown.

the initial data are

$$f(x) = \begin{cases} 1 - |x-4| & \text{if } |x-4| \le 1 \\ 0 & \text{if } |x-4| > 1 \end{cases}$$

$$g(x) = 0$$

so the coefficients evaluate to (Problem 16.10)

$$\begin{aligned} A_n &= \frac{2}{8} \int_0^8 f(x) \sin \frac{n\pi x}{8}\, dx = \frac{32}{n^2\pi^2} \sin \frac{n\pi}{2} \left(1 - \cos \frac{n\pi}{8} \right) \\ B_n &= \frac{2}{n\pi c} \int_0^8 g(x) \sin \frac{n\pi x}{8}\, dx = 0. \end{aligned}$$

The general solution becomes

$$u(x,t)=\sum_{n=1}^{\infty}\frac{32}{n^2\pi^2}\sin\frac{n\pi}{2}\left(1-\cos\frac{n\pi}{8}\right)\cos\frac{n\pi ct}{8}\sin\frac{n\pi x}{8}.$$

Notice that because of the $\sin(n\pi/2)$ factor, the summand is zero when n is even. The series can be made to converge faster, then, by summing over the odd integers only. Substitute $n = 2m-1$.

$$u(x,t)=\sum_{2m-1=1}^{\infty}\frac{32}{(2m-1)^2\pi^2}\underbrace{\sin\frac{(2m-1)\pi}{2}}_{=(-1)^{m-1}}\left[1-\cos\frac{(2m-1)\pi}{8}\right]\cos\frac{(2m-1)\pi ct}{8}\sin\frac{(2m-1)\pi x}{8}.$$

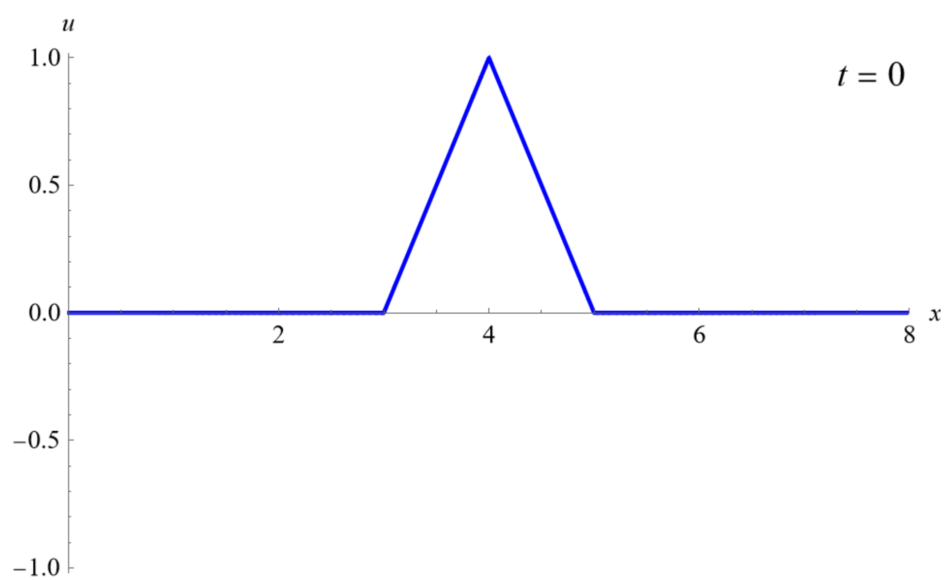
Therefore,

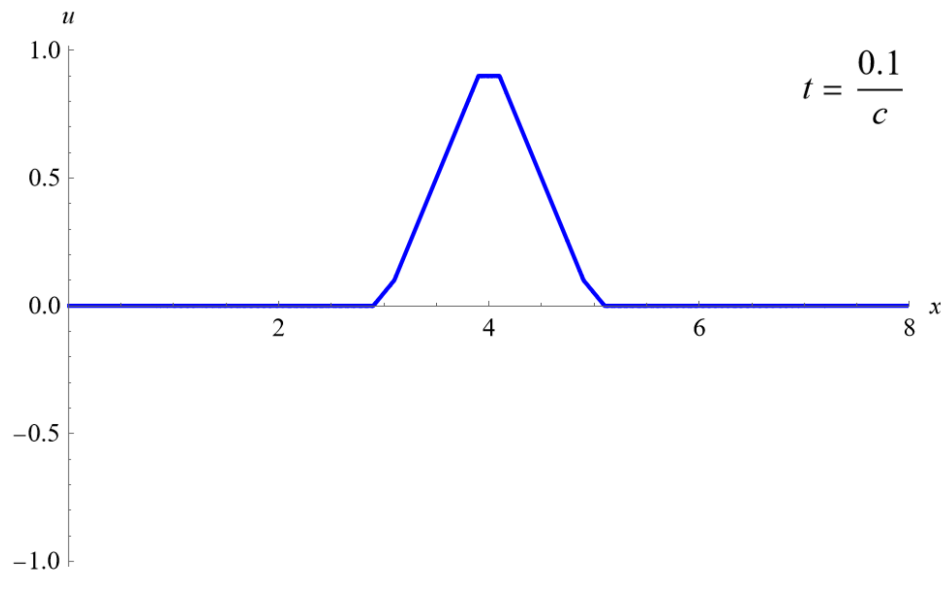
$$u(x,t)=\frac{32}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\left[1-\cos\frac{(2m-1)\pi}{8}\right]\cos\frac{(2m-1)\pi ct}{8}\sin\frac{(2m-1)\pi x}{8}.$$

The period of the fundamental ($m = 1$) is

$$\tau=\frac{2\pi}{\omega_1}=\frac{2\pi}{\frac{\pi c}{8}}=\frac{16}{c}.$$

Below are 69 plots of $u(x,t)$ versus x at various times from $t = 0$ to $t = \tau$ in order to illustrate the solution's behavior. The first 1000 terms are used in the series for each graph.

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2
4
6
8
x
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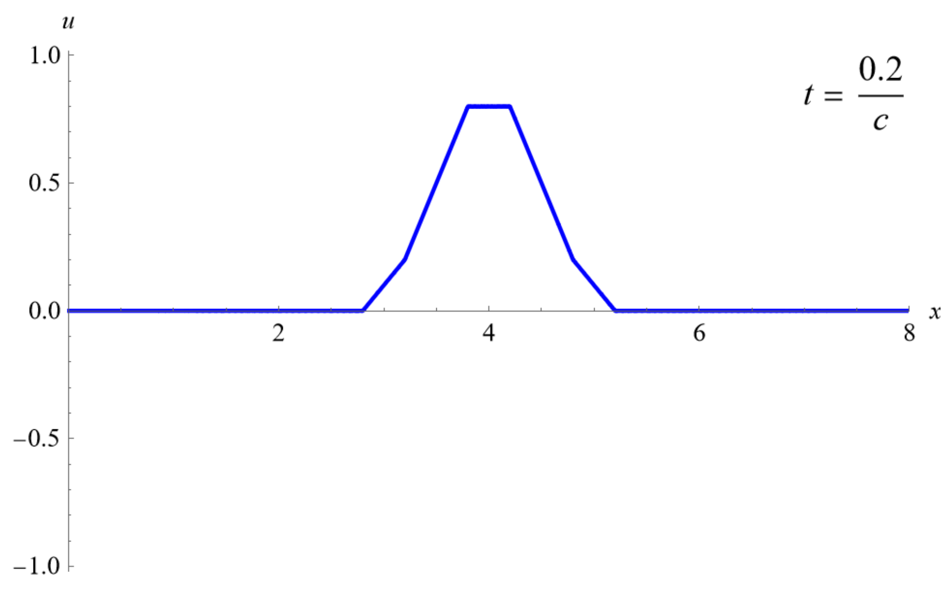
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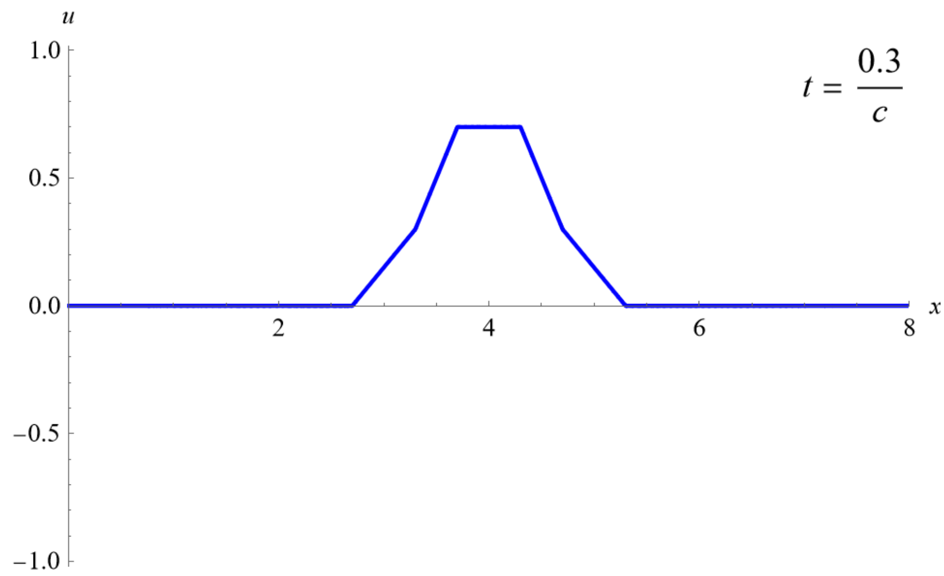
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x
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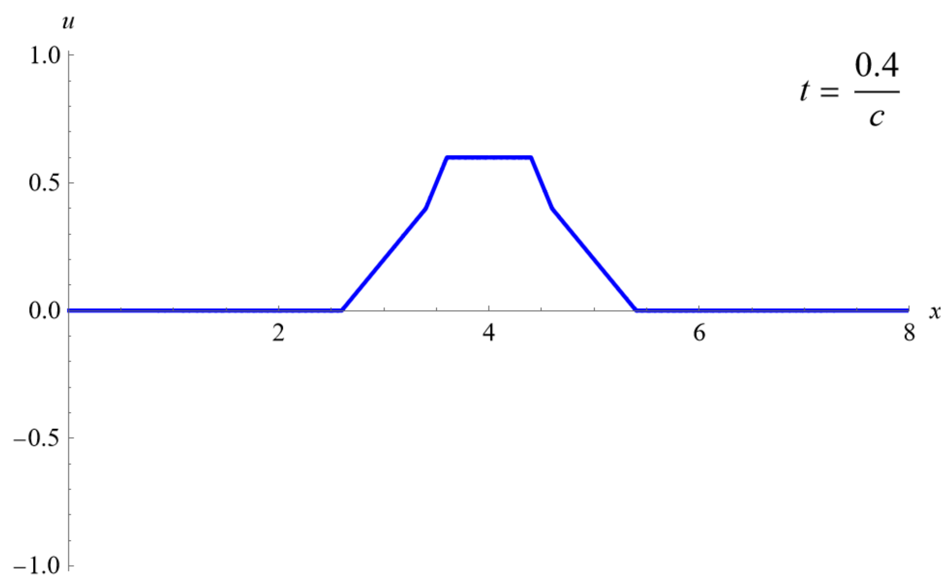
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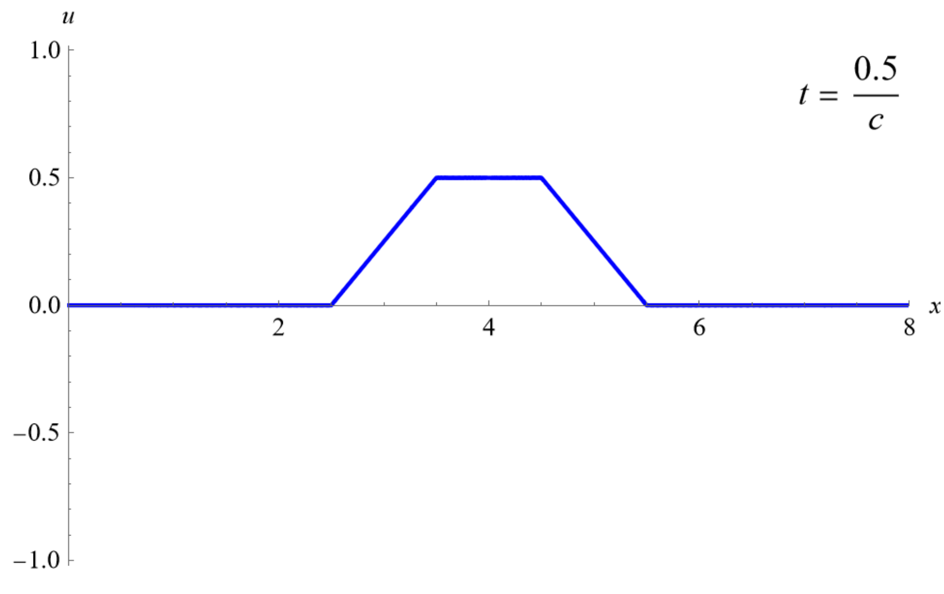
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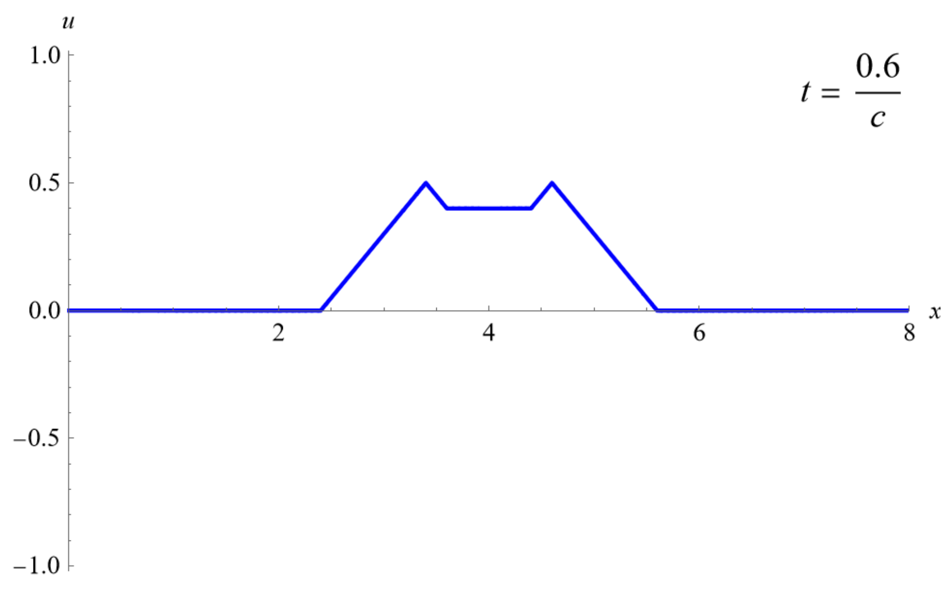
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4
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x
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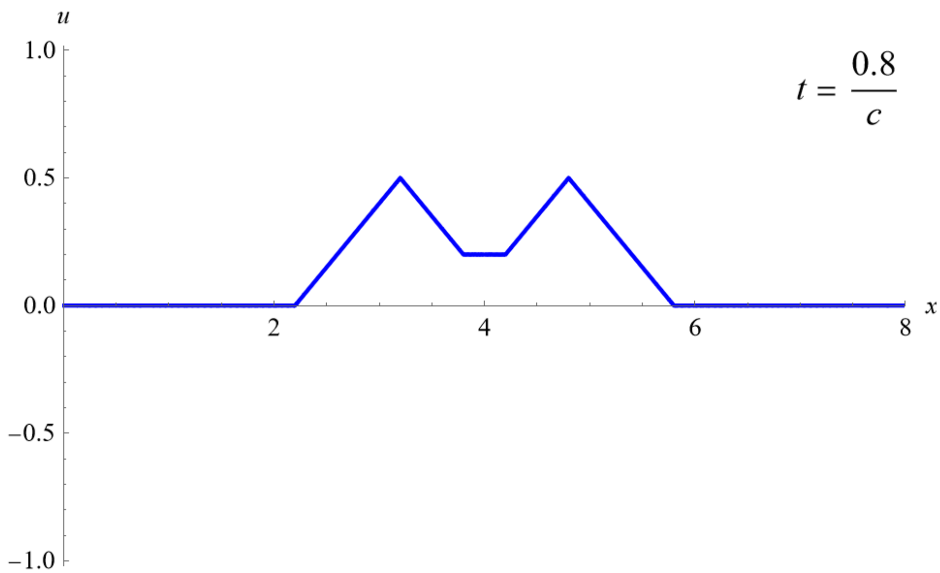

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4
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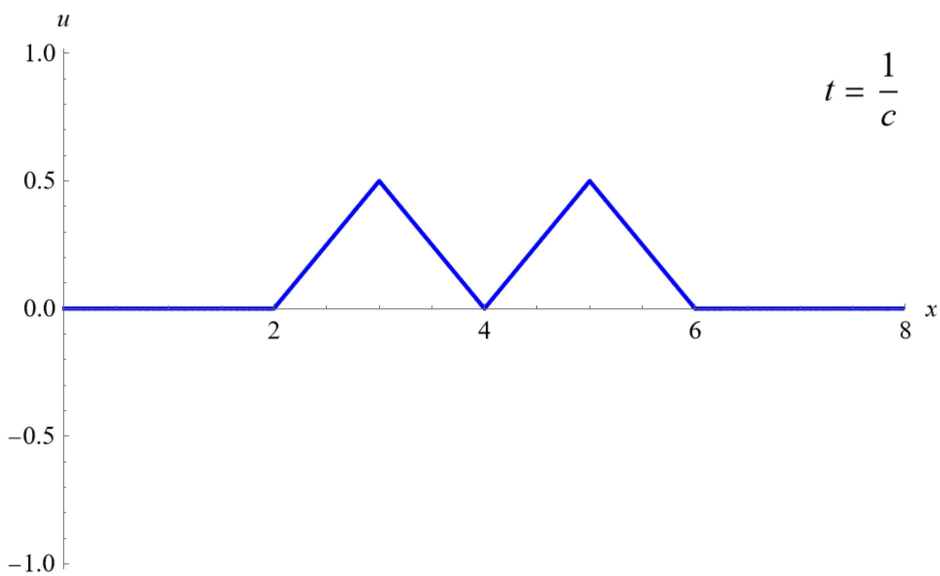
8
x
t = 0.8/c



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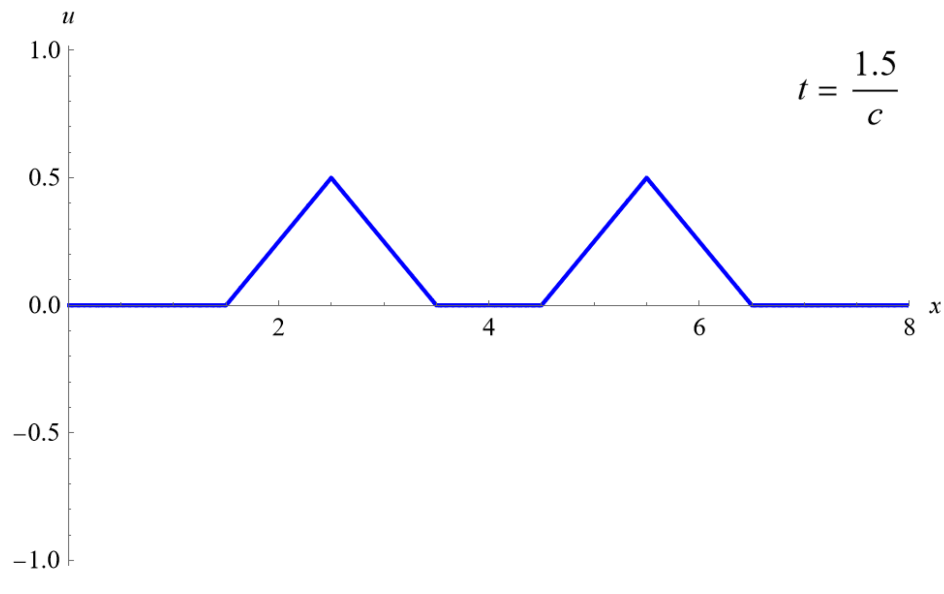
x
t = 1/c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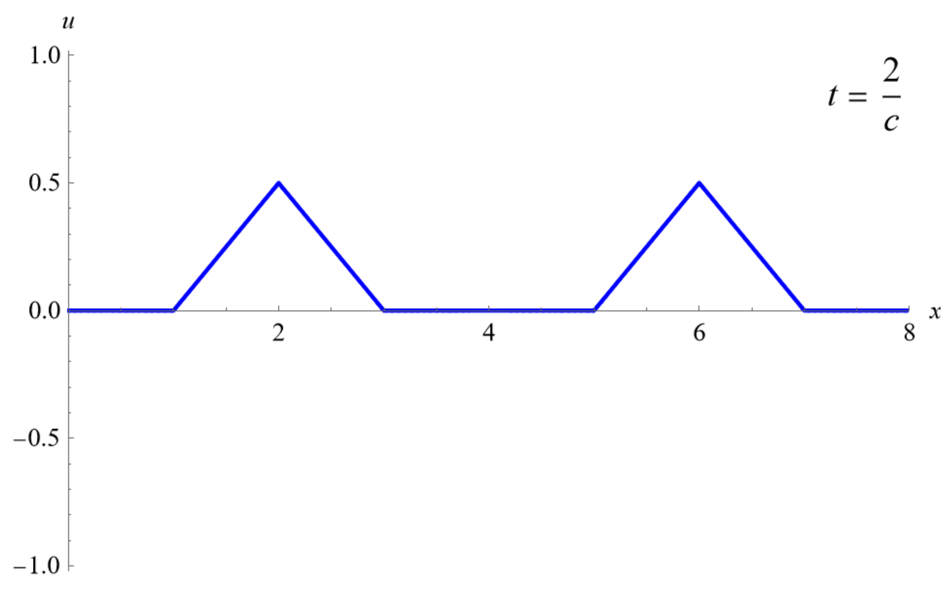
x
t = 1.5/c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6
8
x
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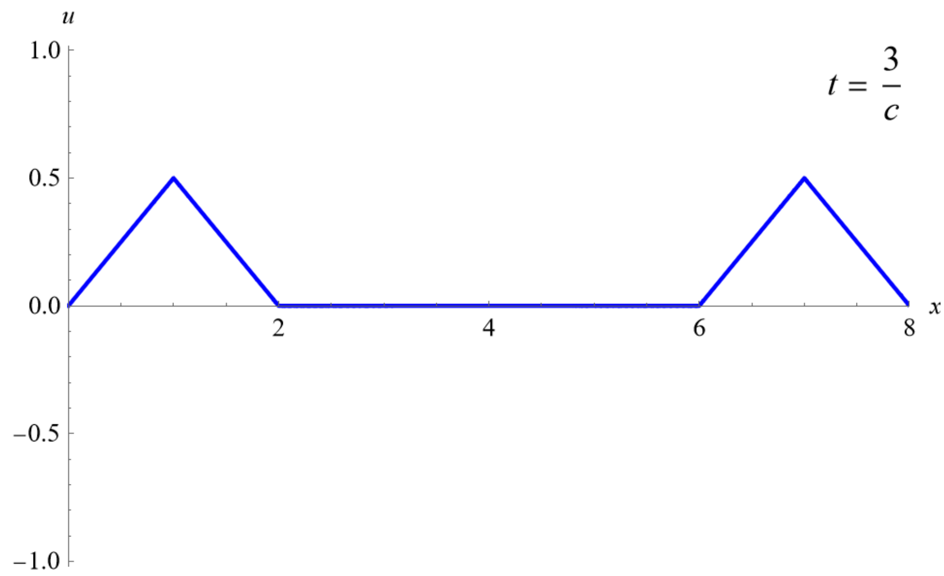
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8
x
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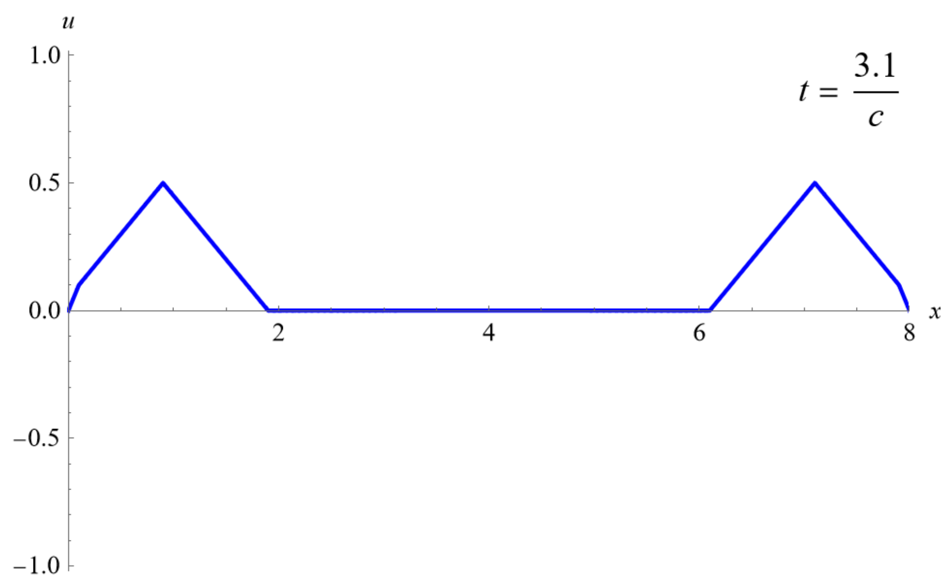
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x
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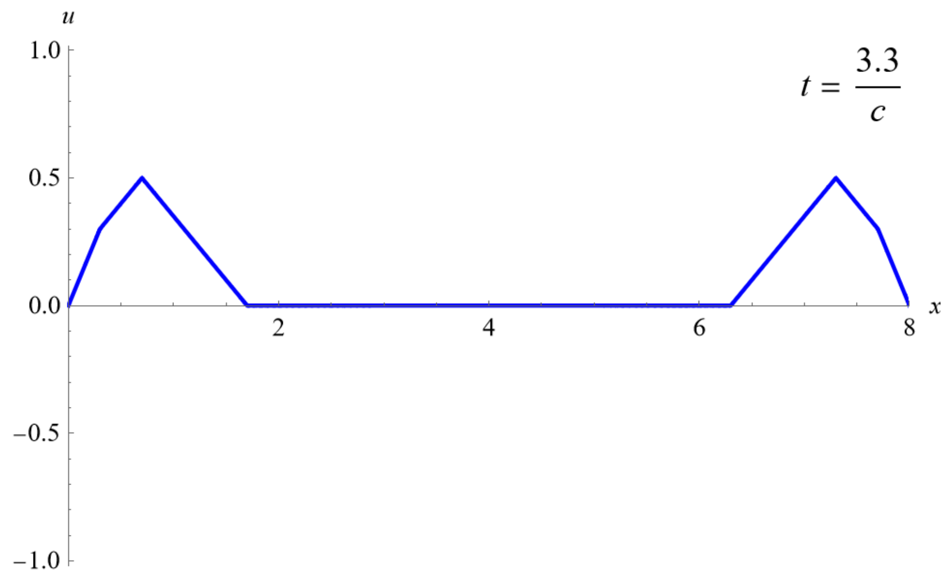
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8
x
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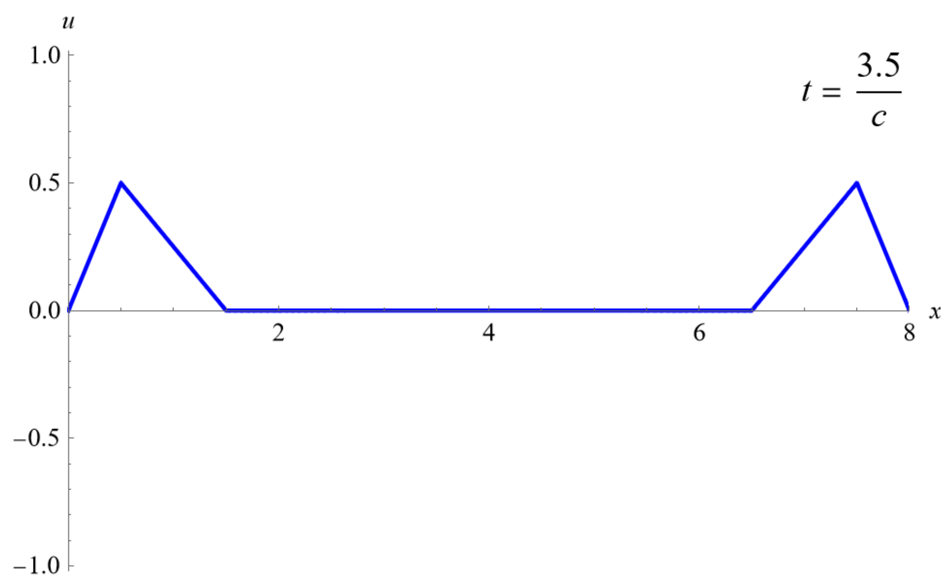
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8
x
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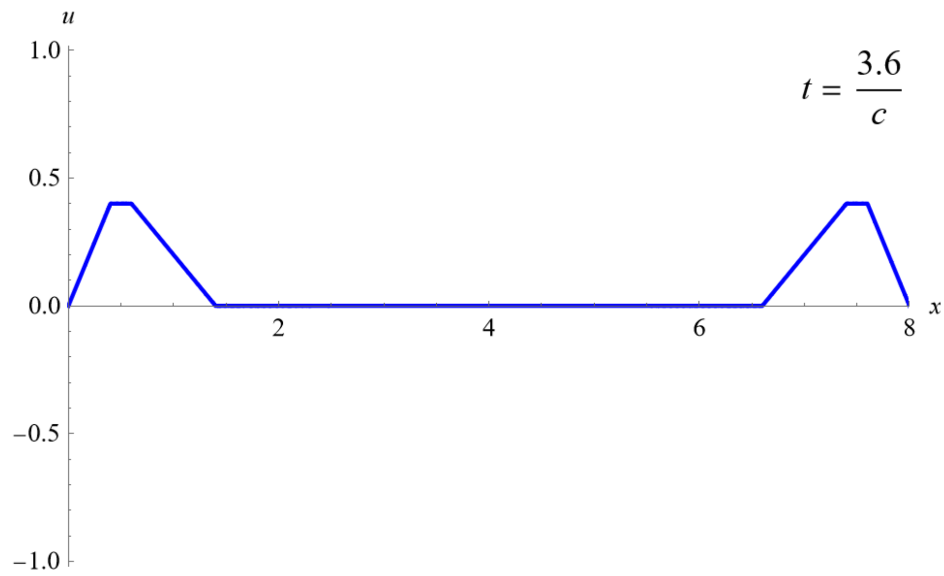
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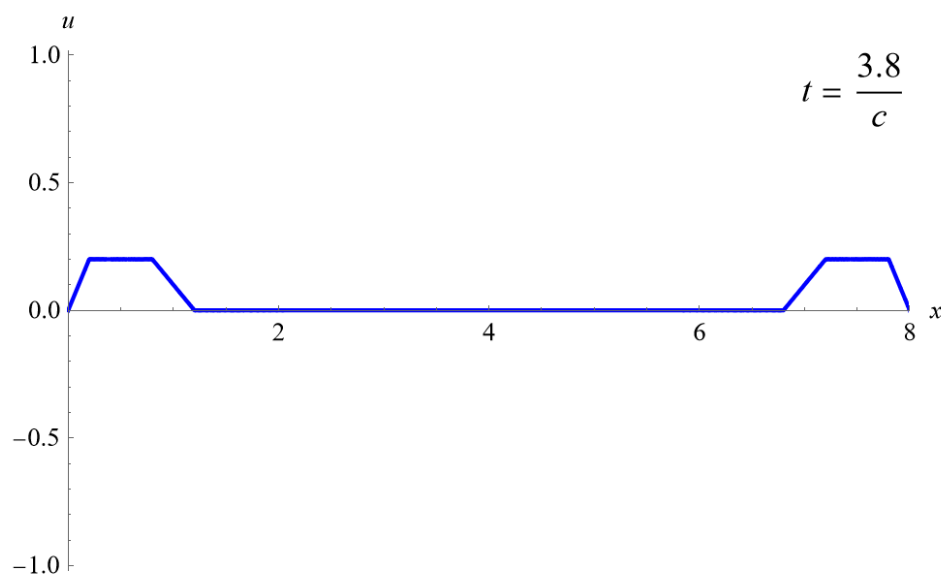
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x
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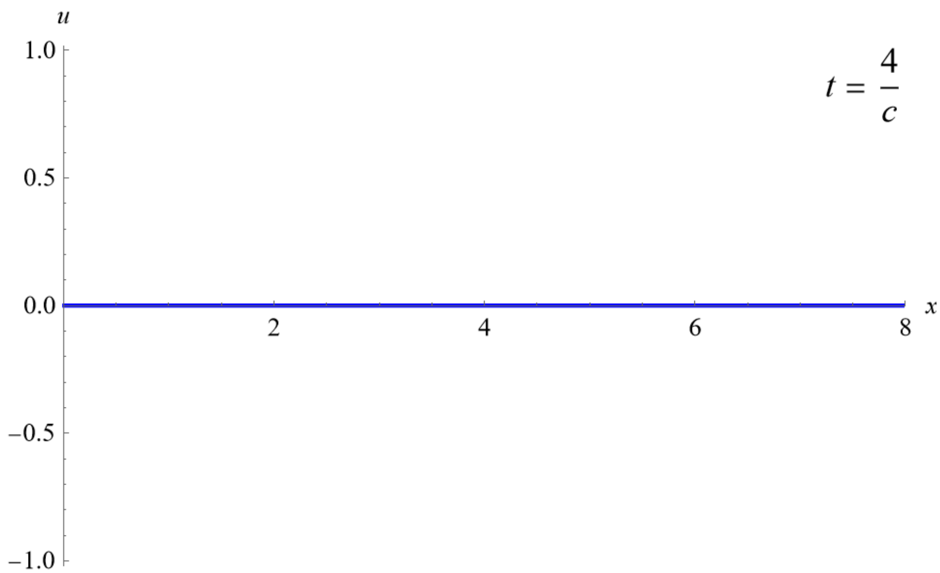
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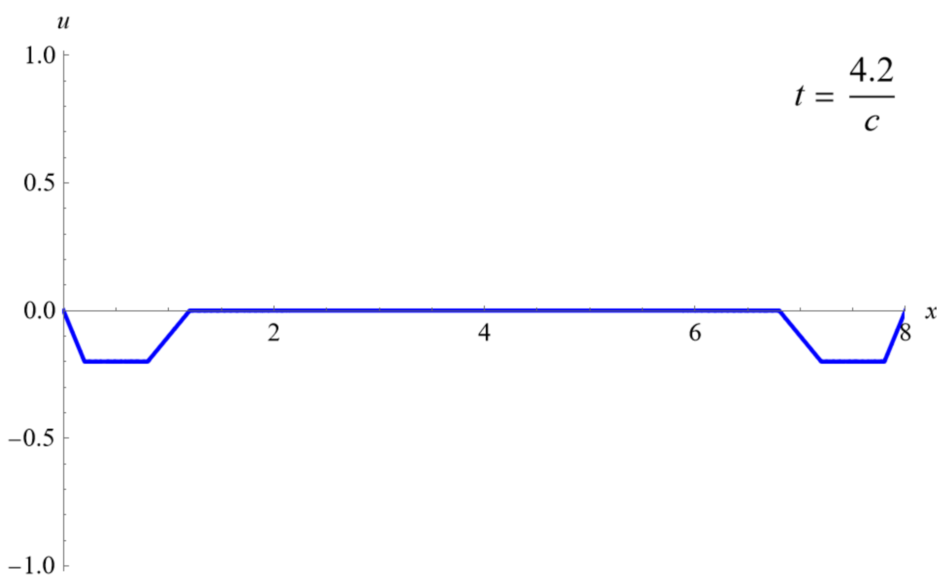
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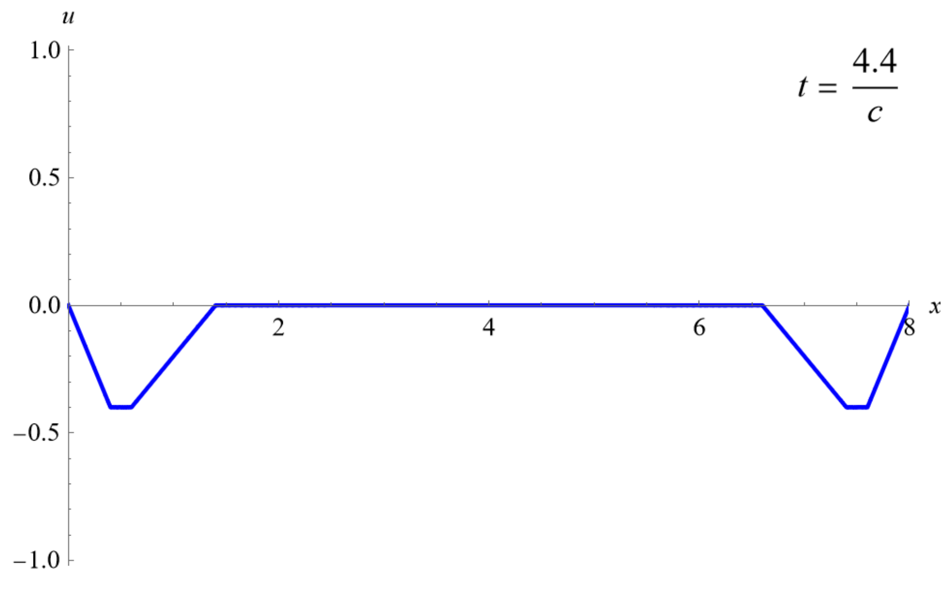
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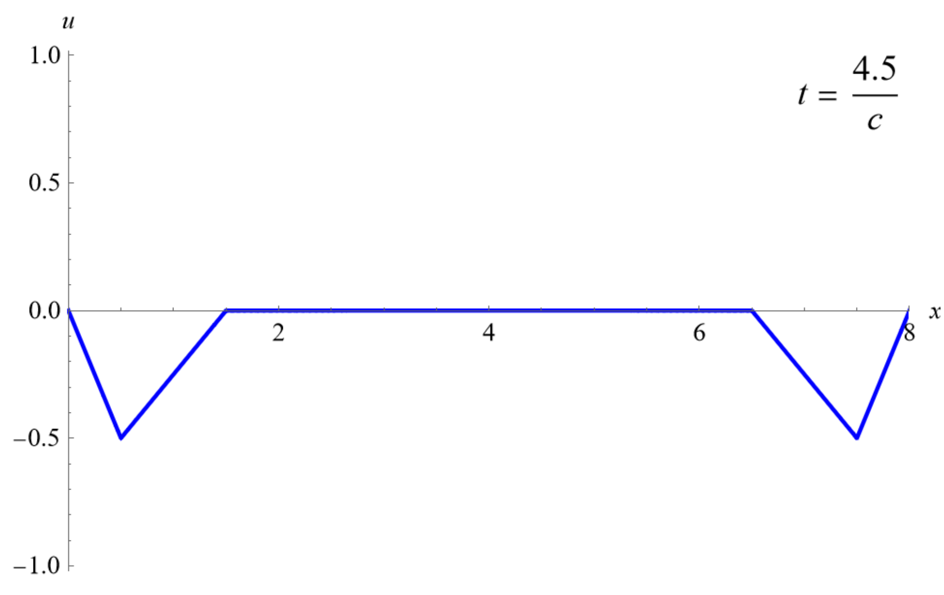
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x
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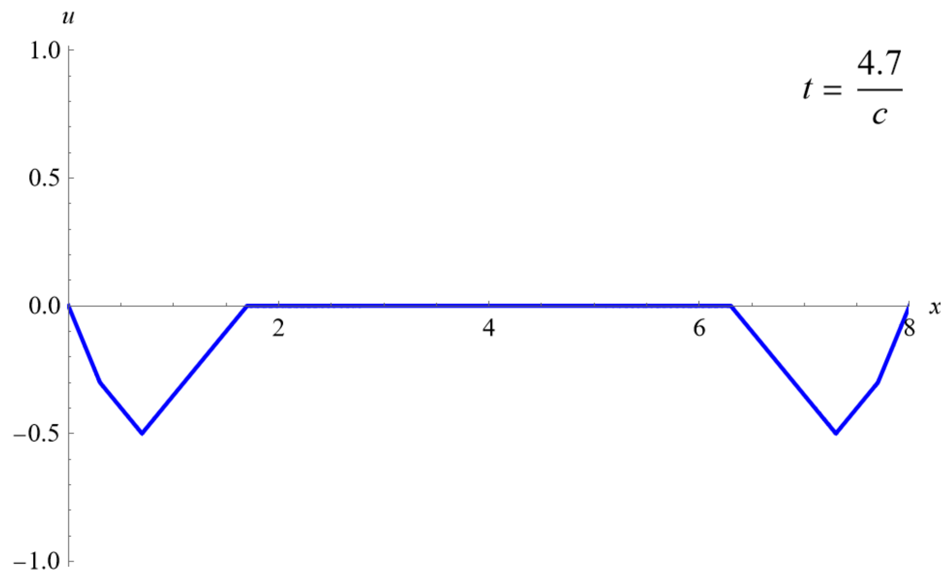
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x
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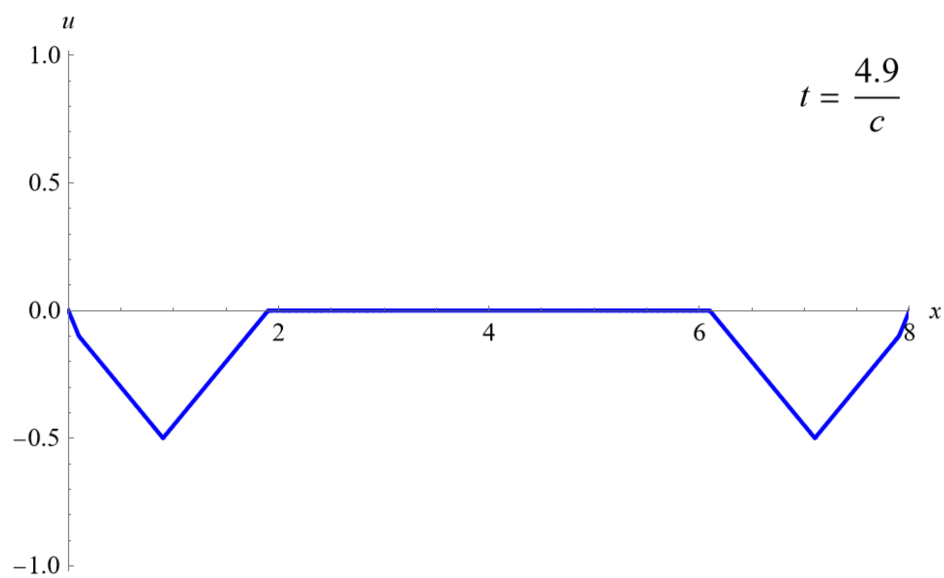
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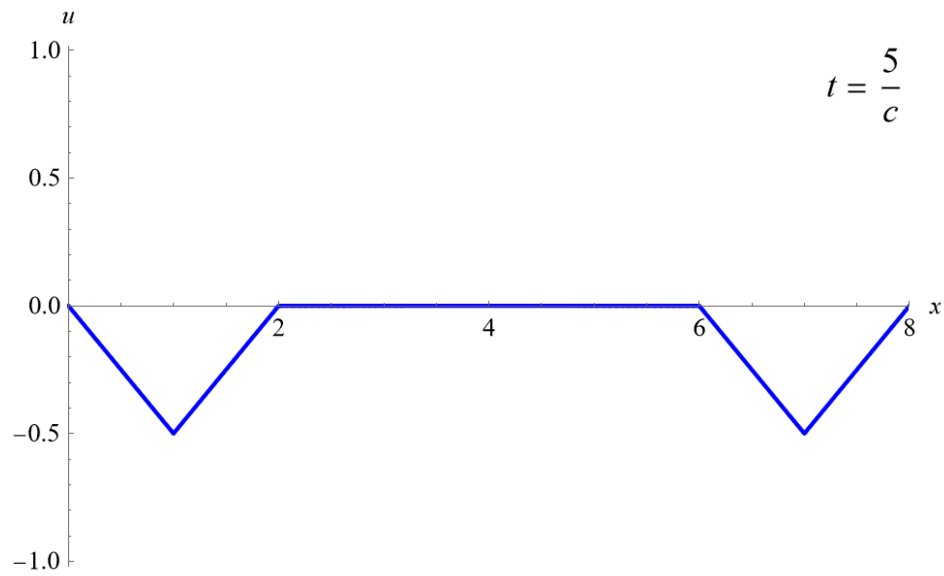
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8
x
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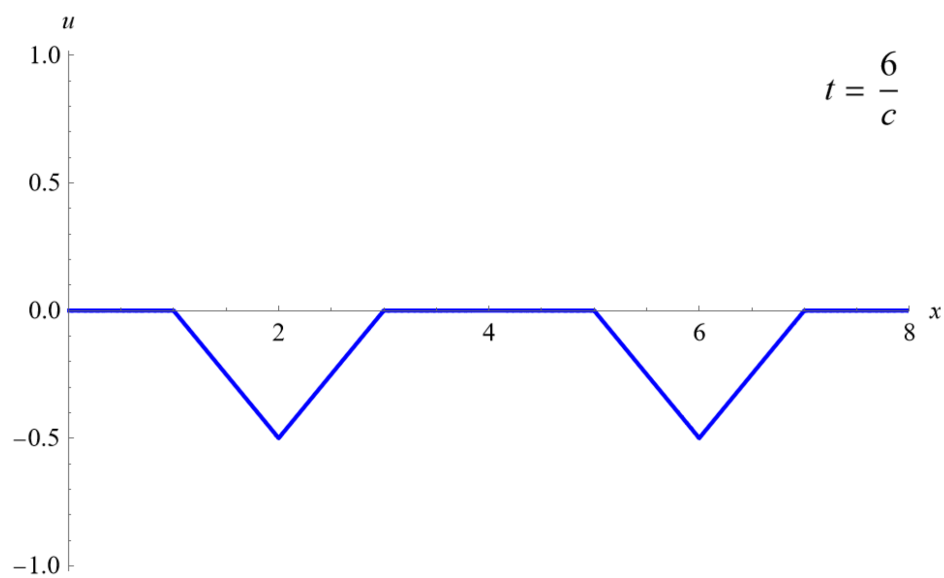
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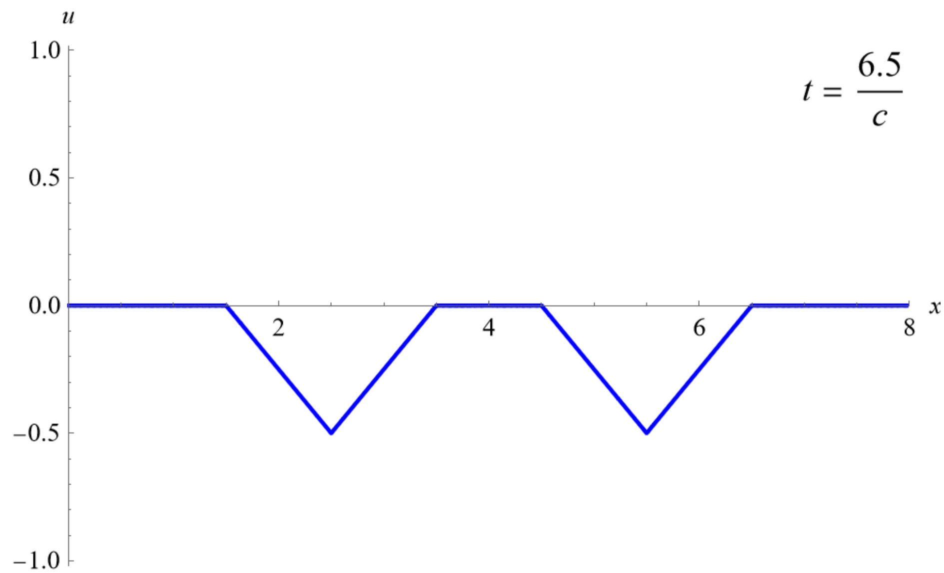
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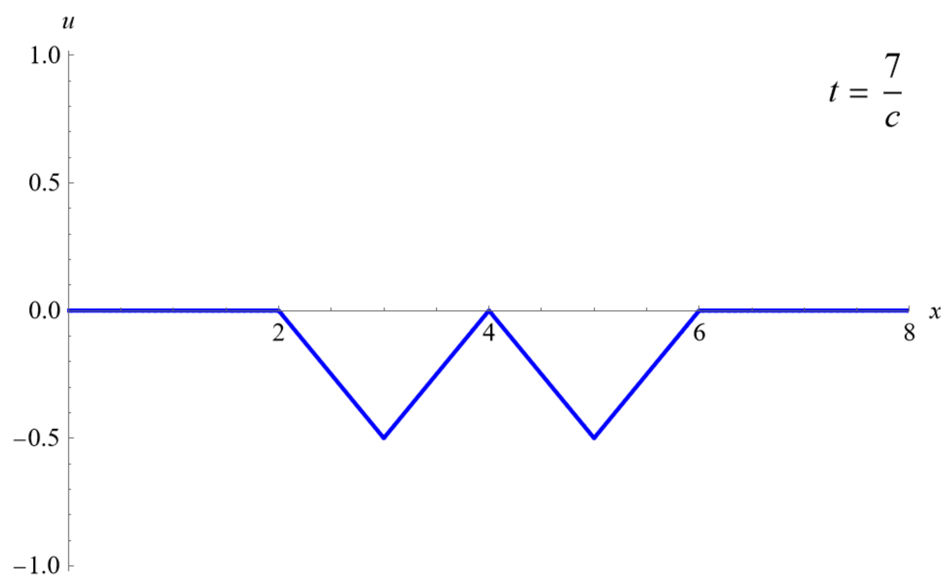
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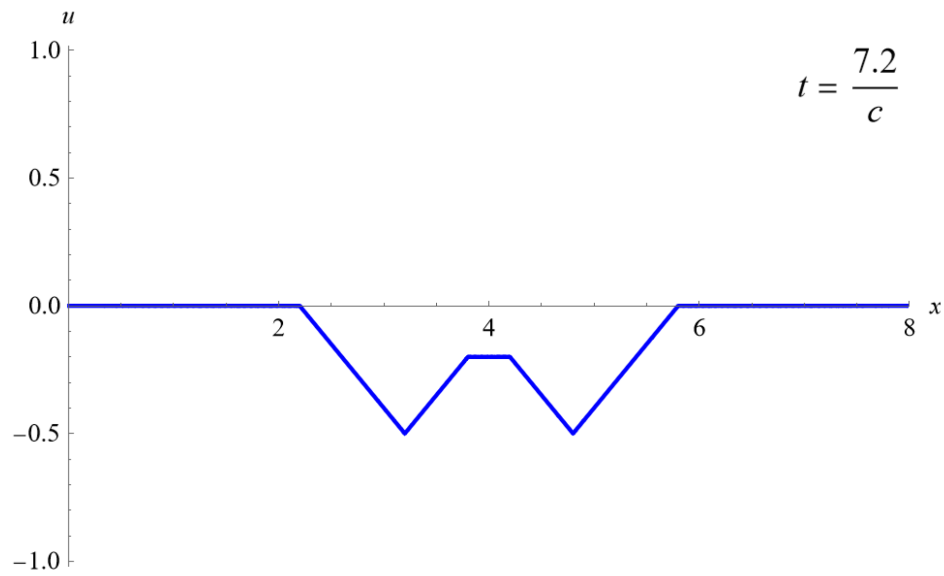
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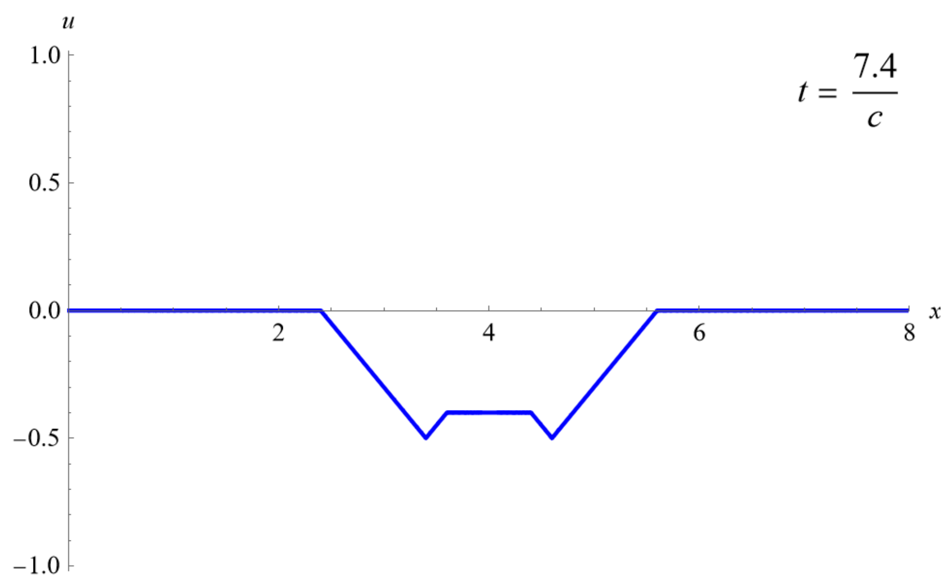
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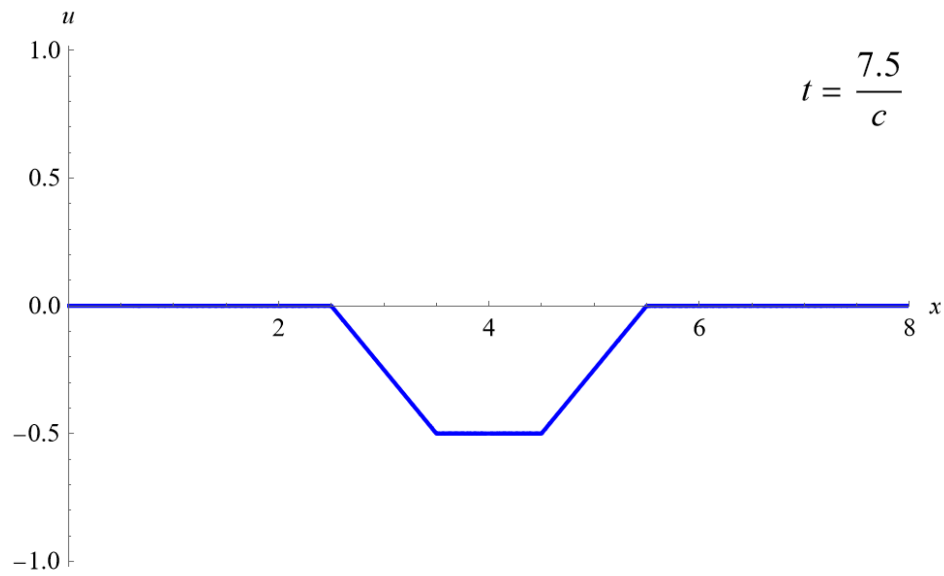
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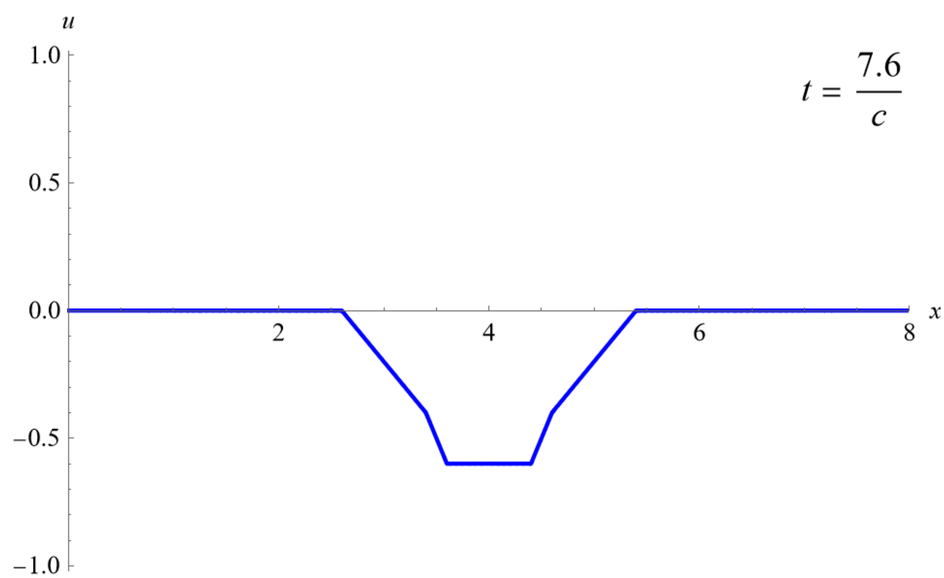
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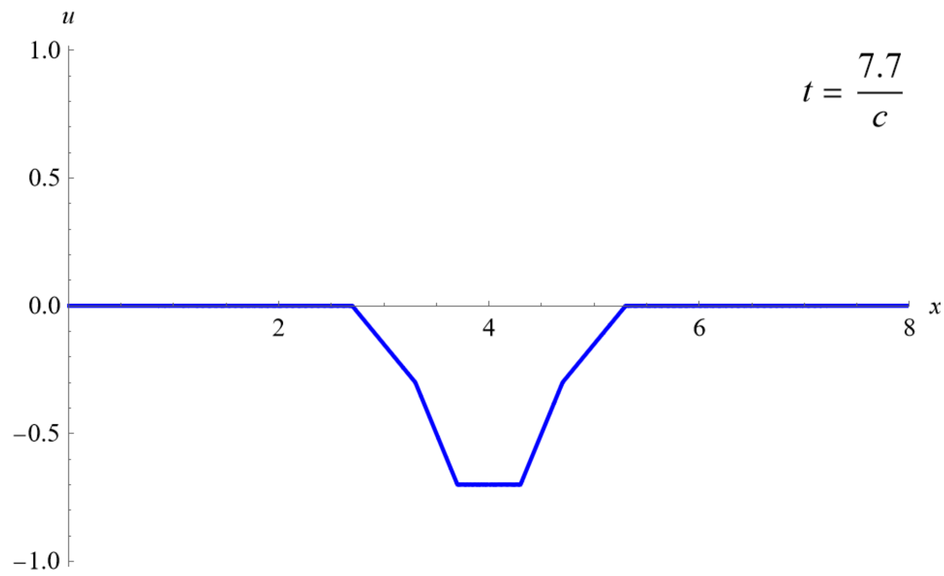
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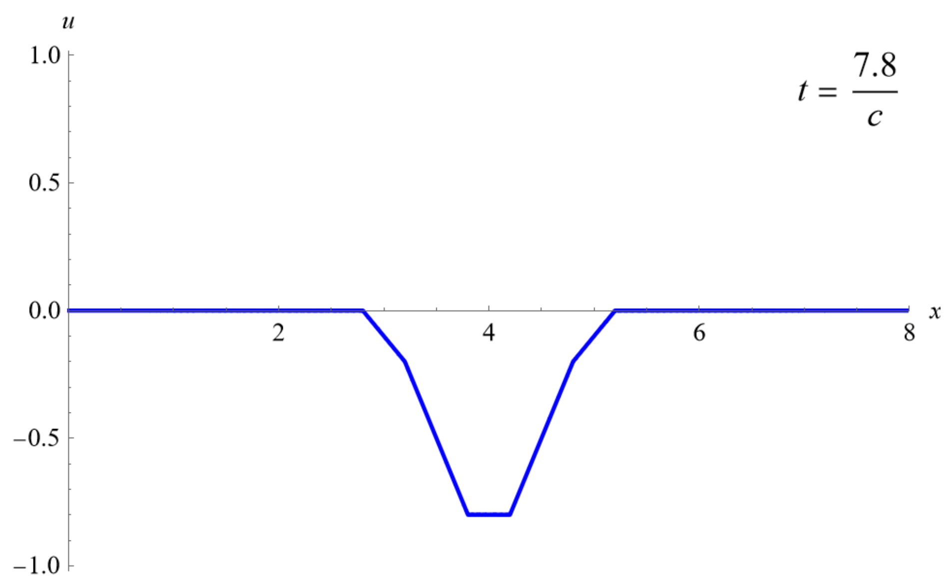
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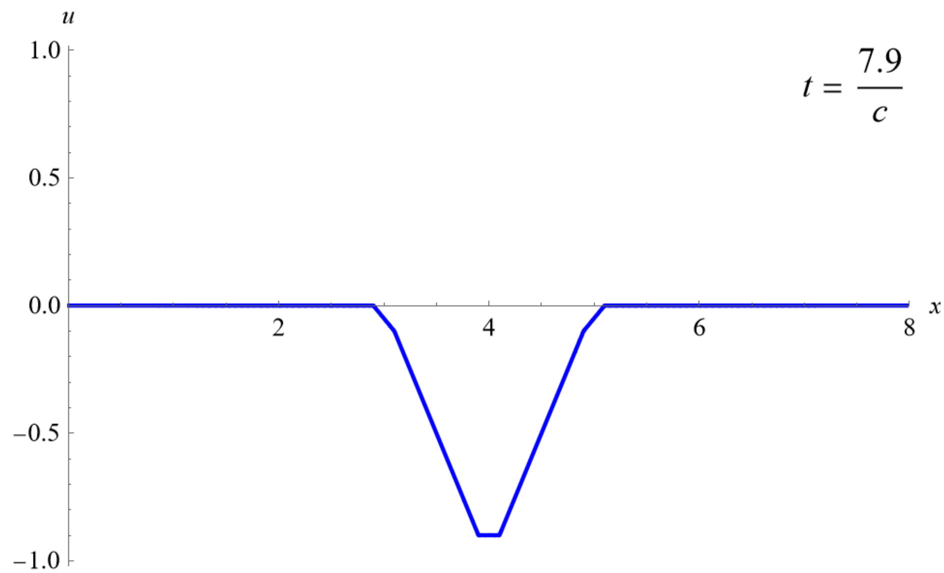
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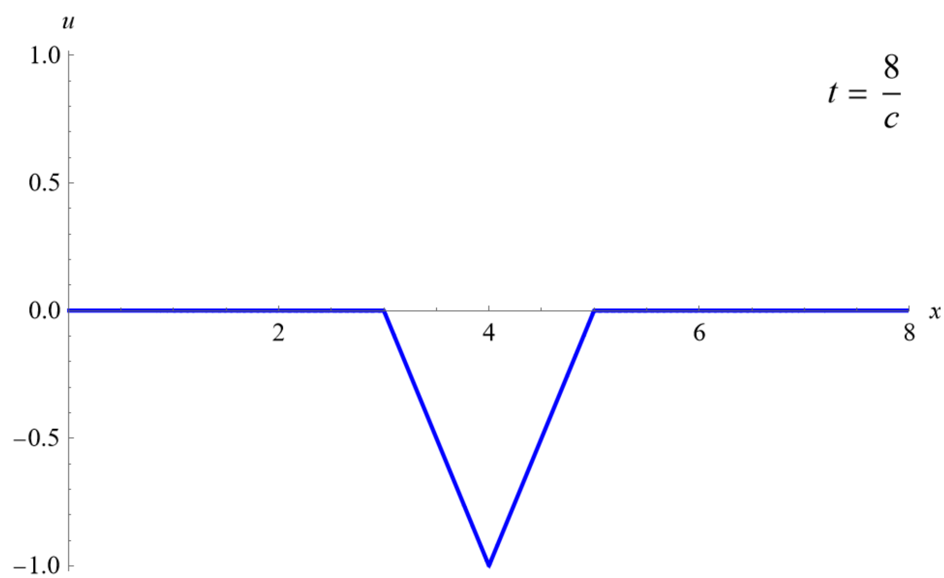
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8
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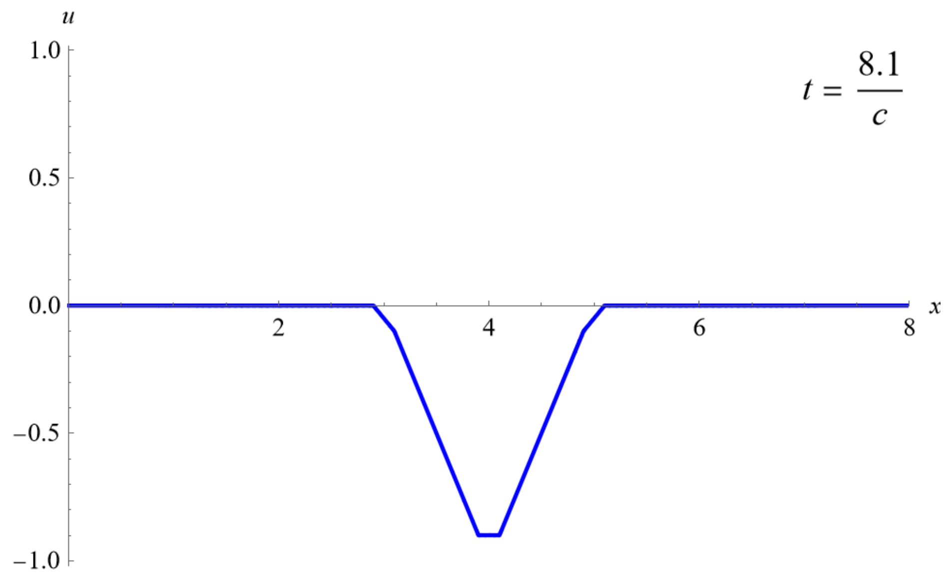
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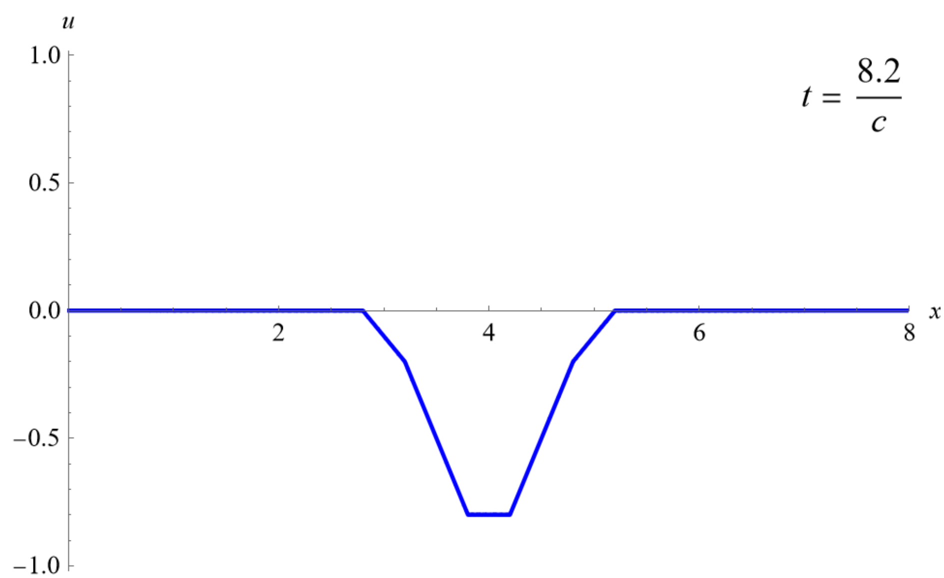
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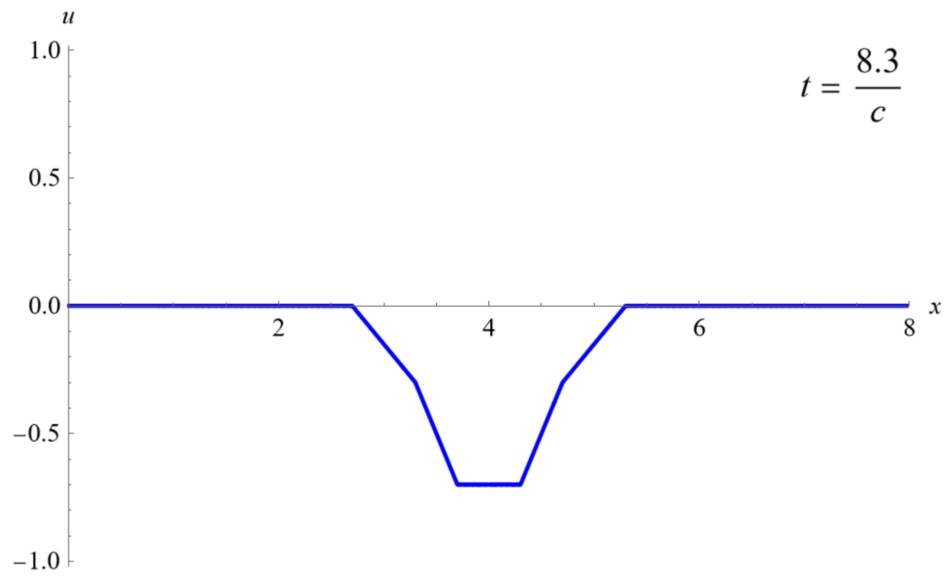
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2
4
6
8
x
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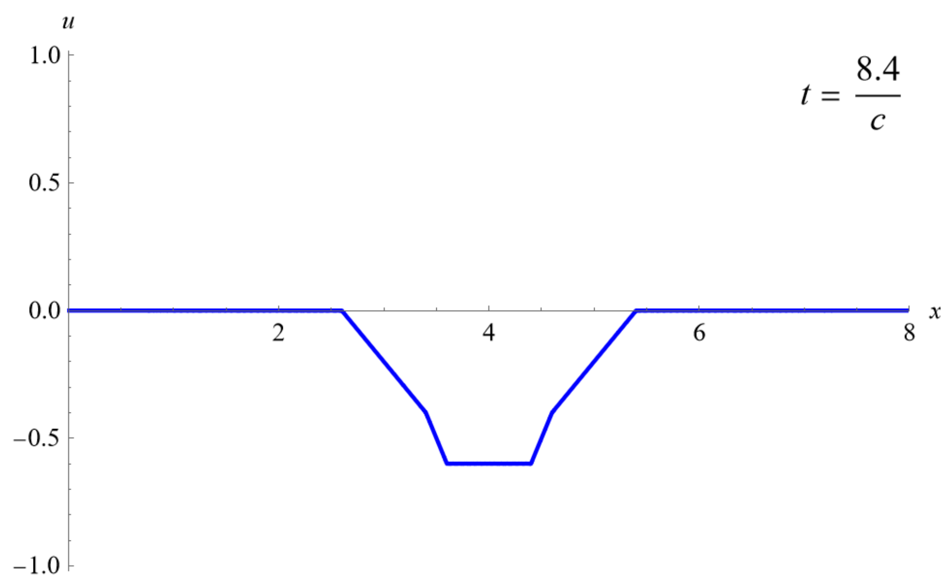
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8
x
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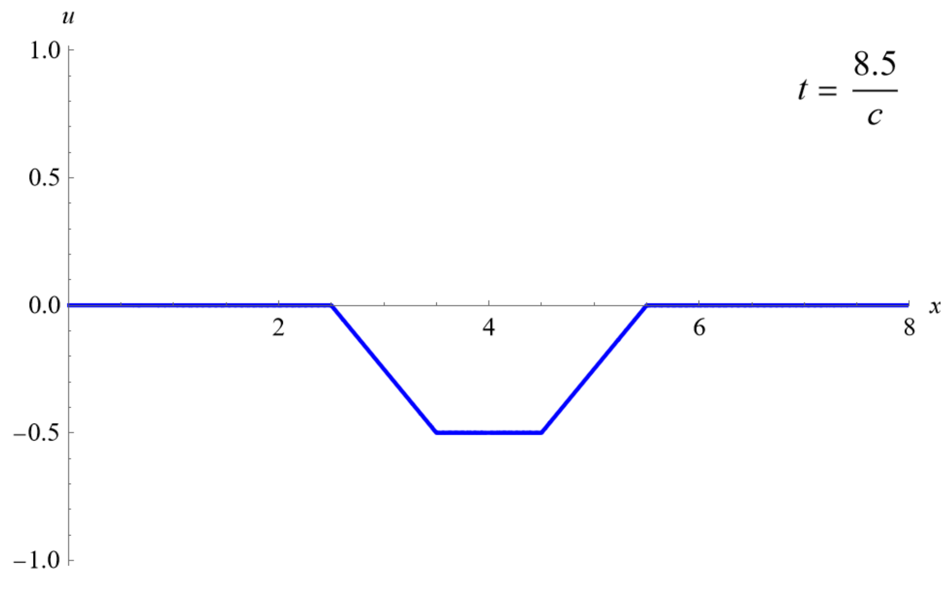
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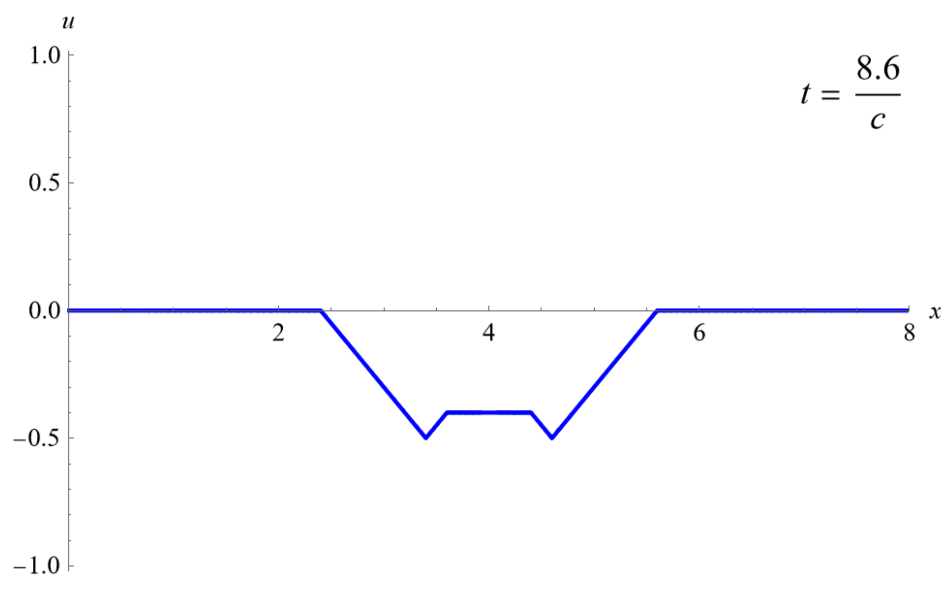
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8
x
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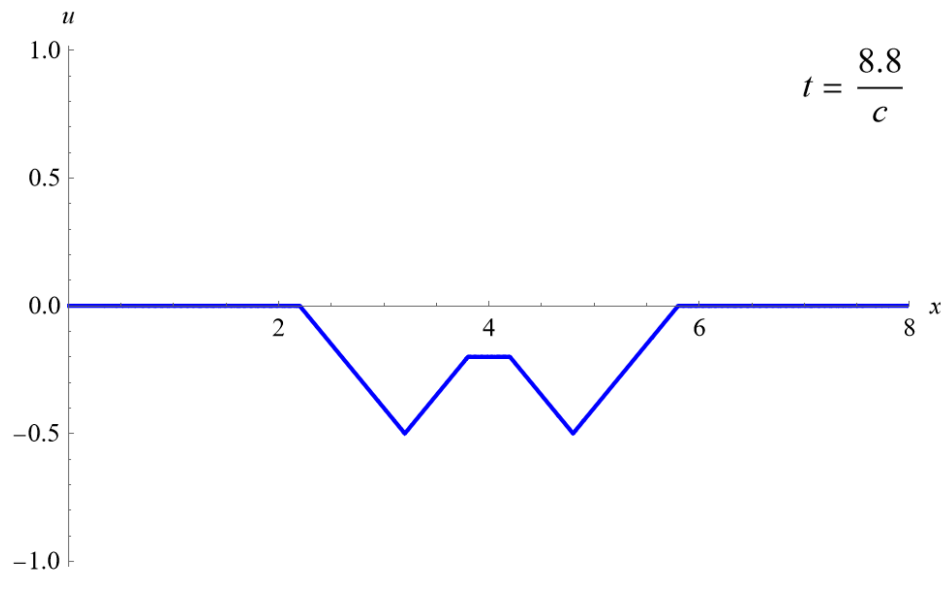
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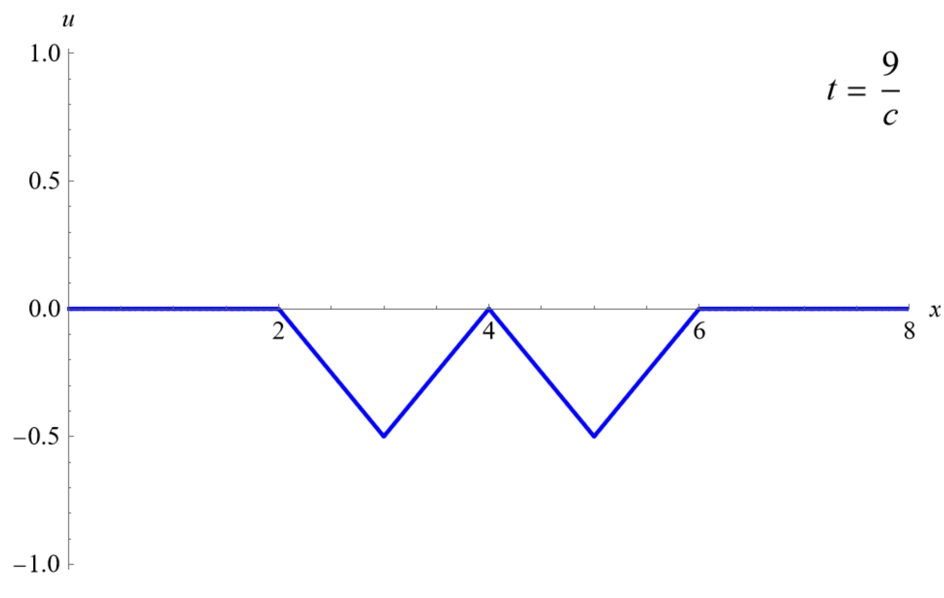
4
6
8
x
t = 8.8/c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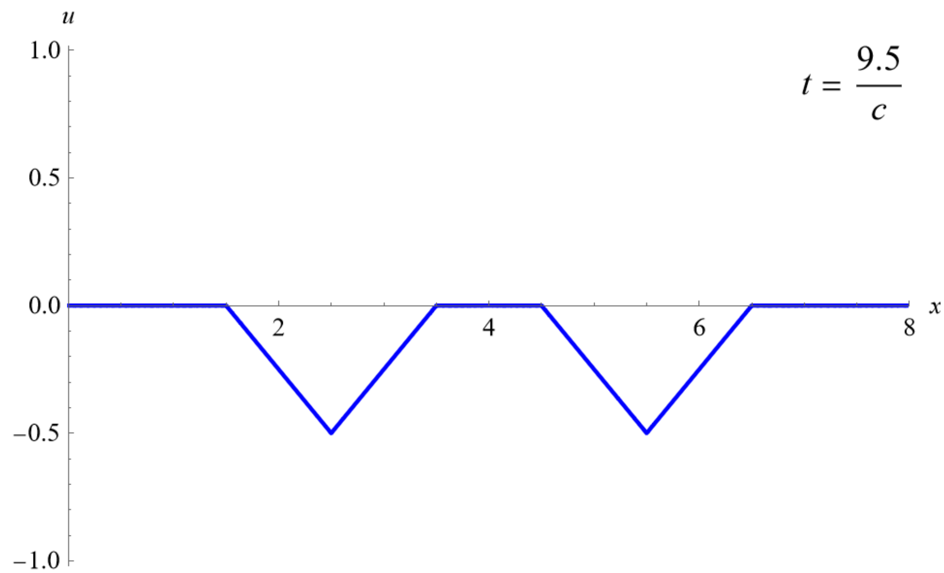
6
8
x
t = 9/c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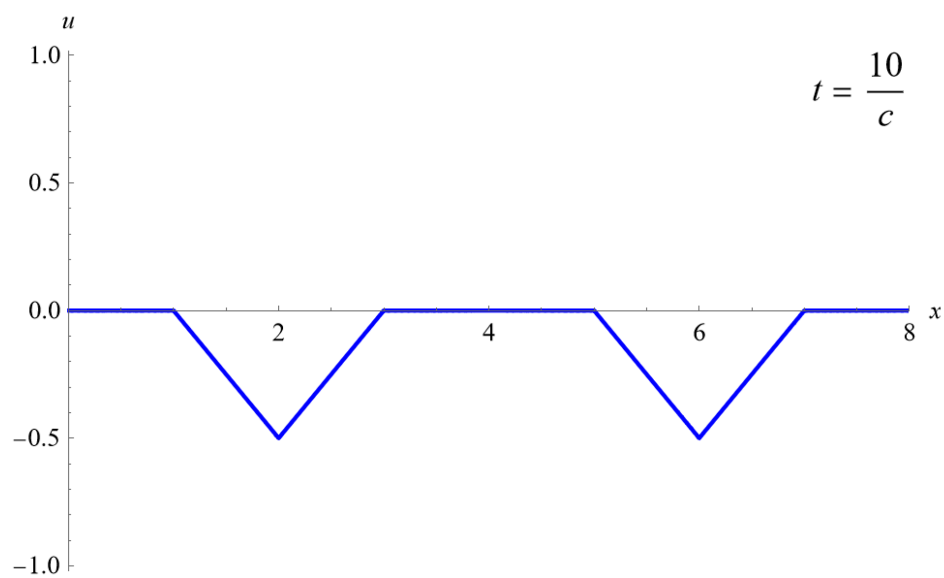
4
6
8
x
$t = \frac{9.5}{c}$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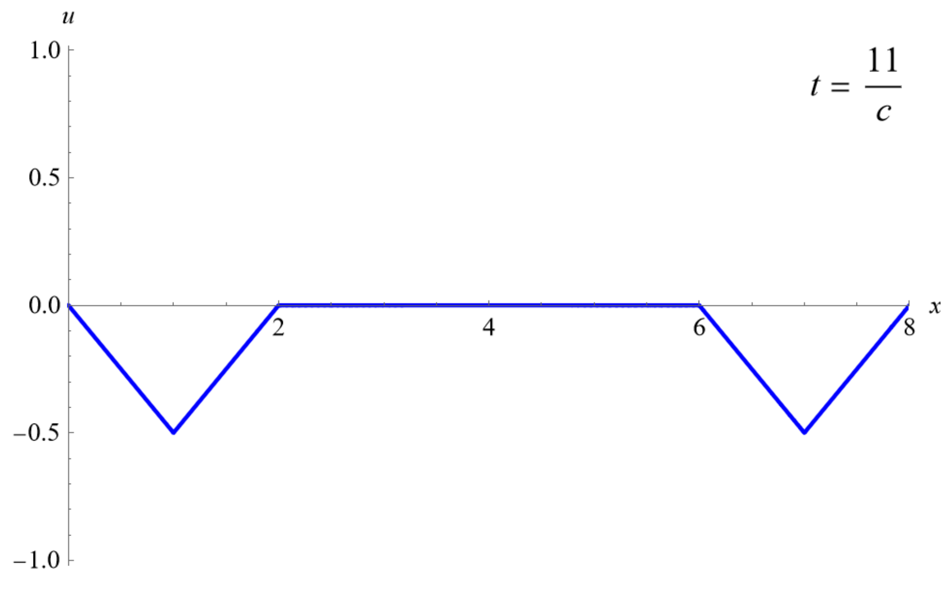
4
6
8
x
$t = \frac{10}{c}$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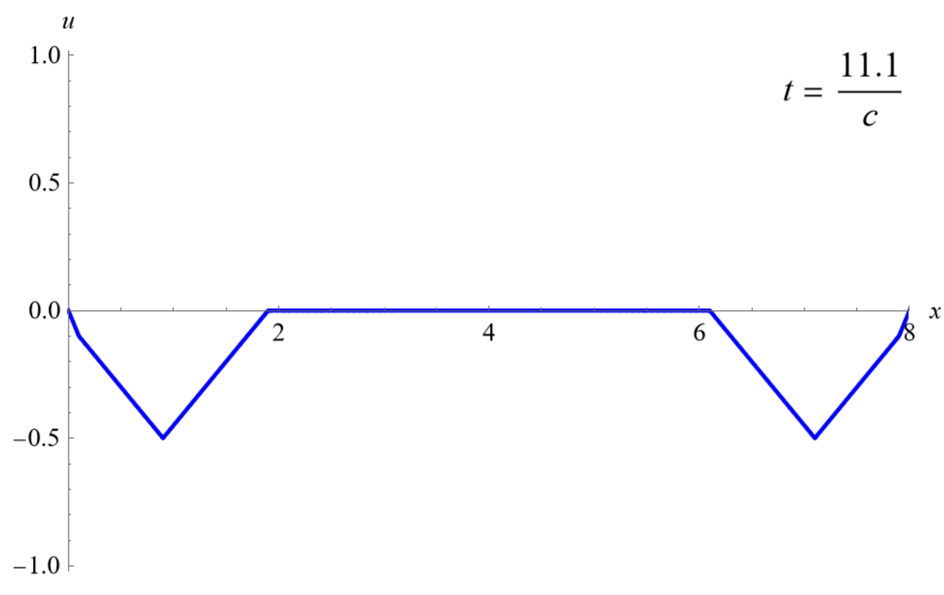
x
$t = \frac{11}{c}$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x
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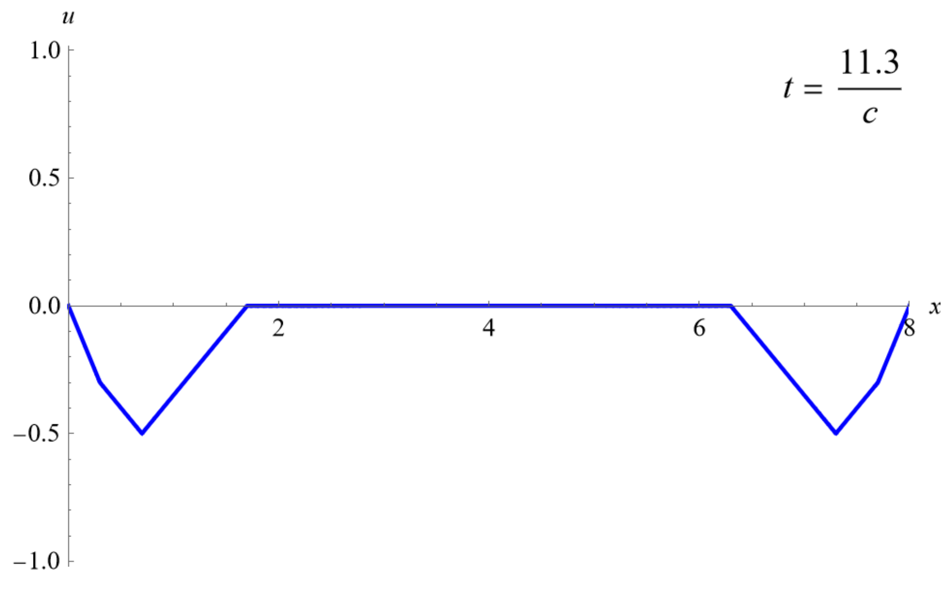
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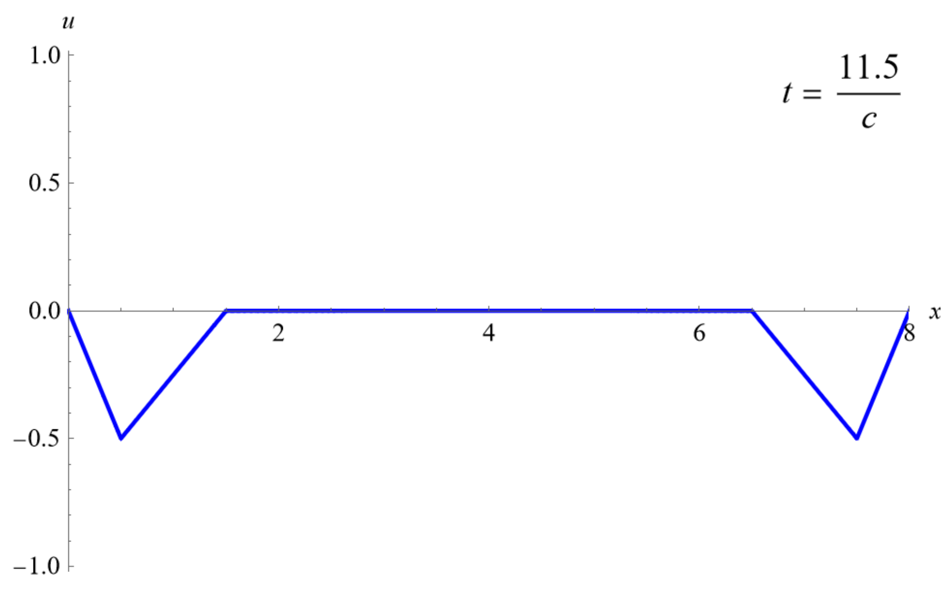
x
$t = \frac{11.3}{c}$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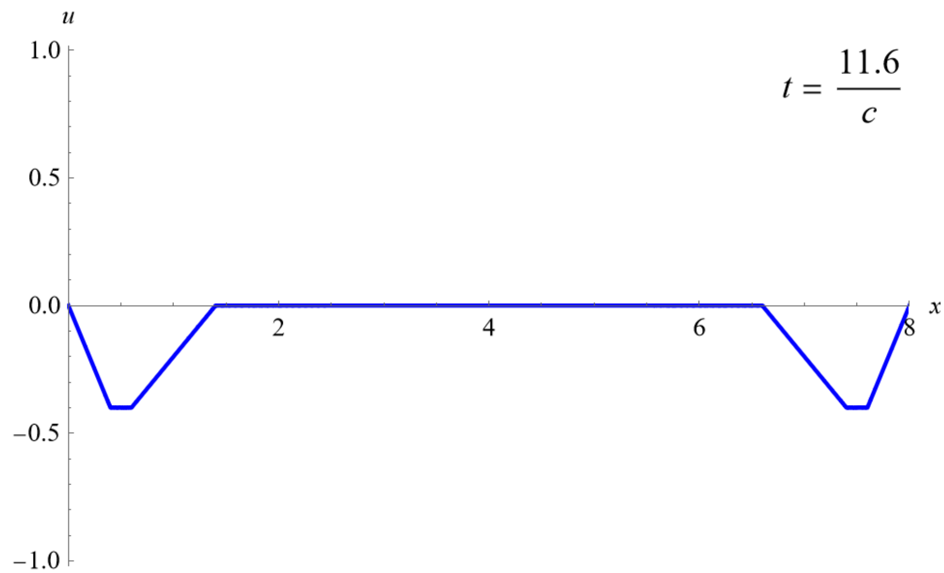
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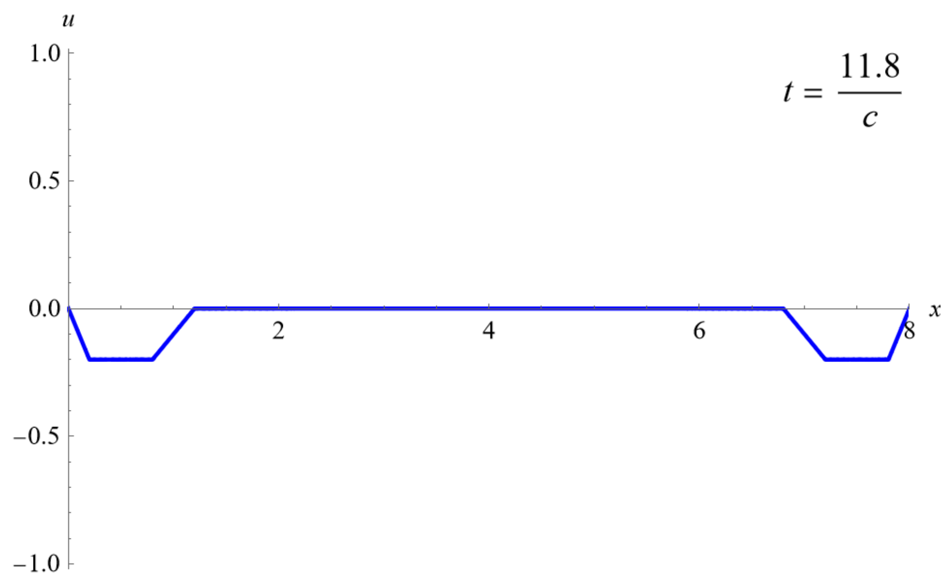
x
t = 11.6/c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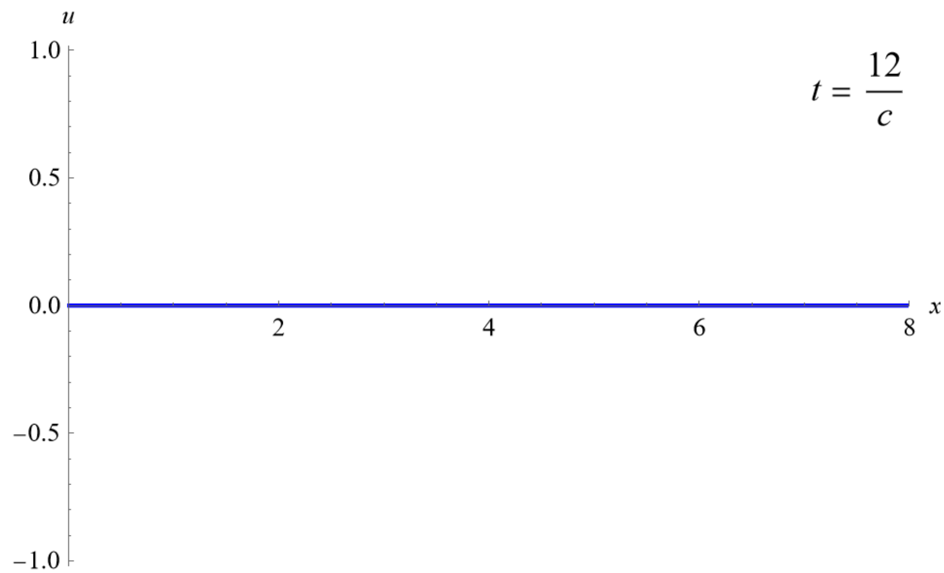
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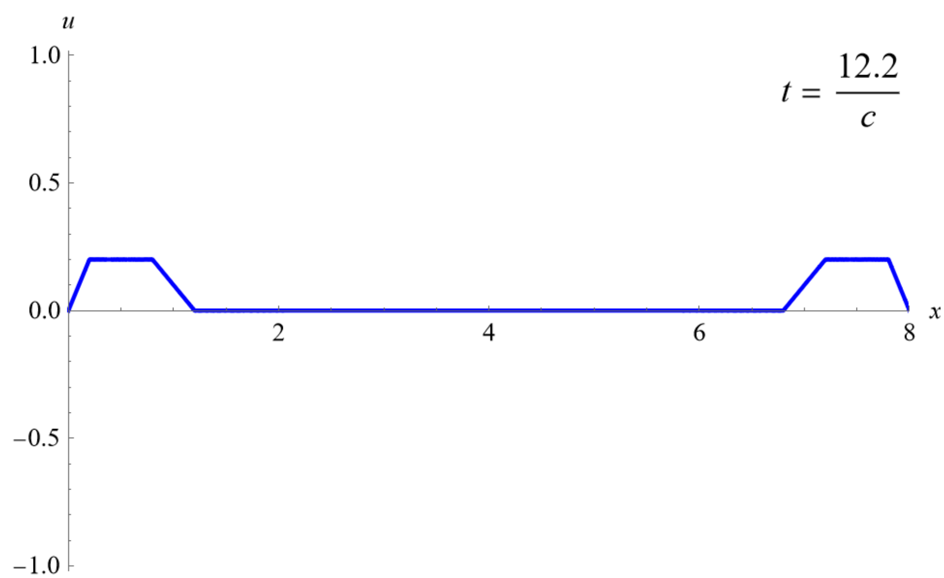
4
6
8
x
t = 12/c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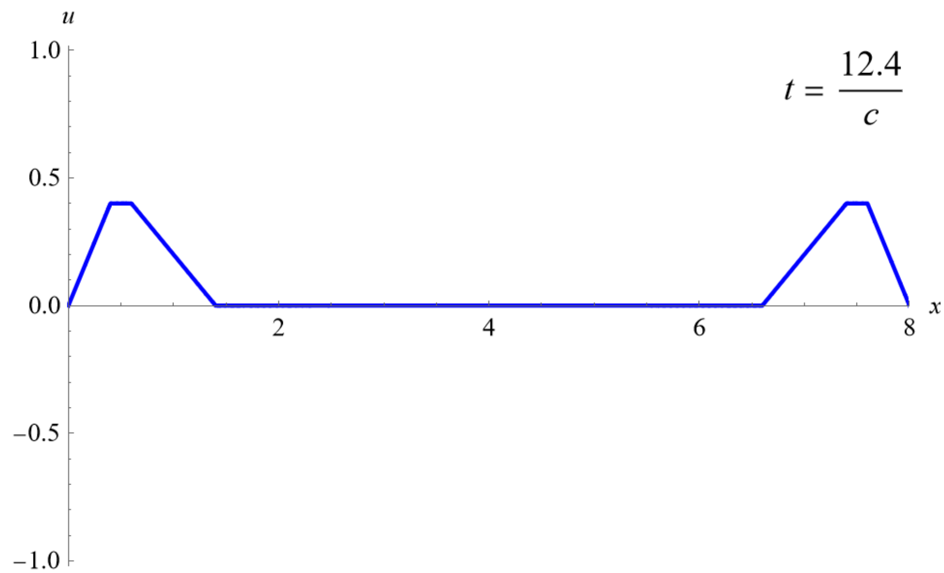
t = 12.2/c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x
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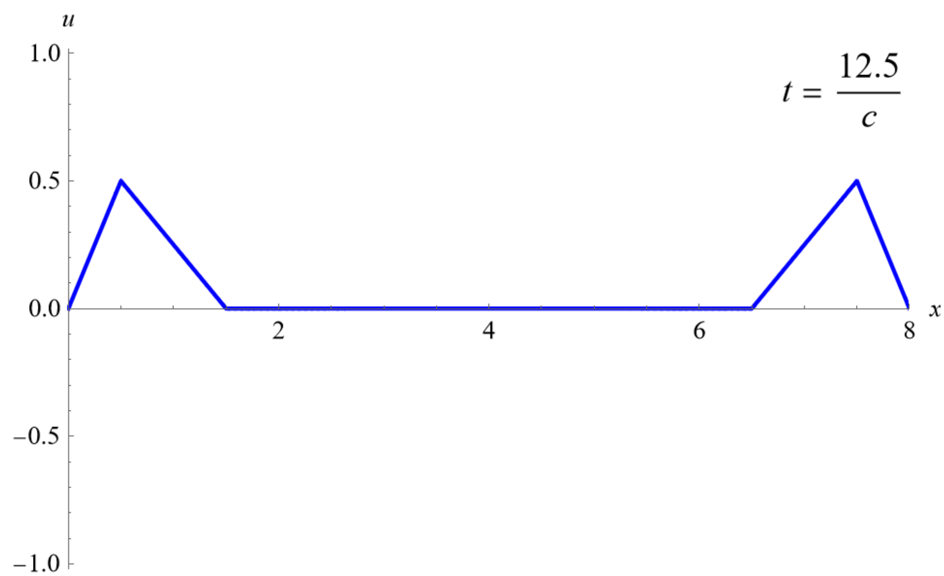
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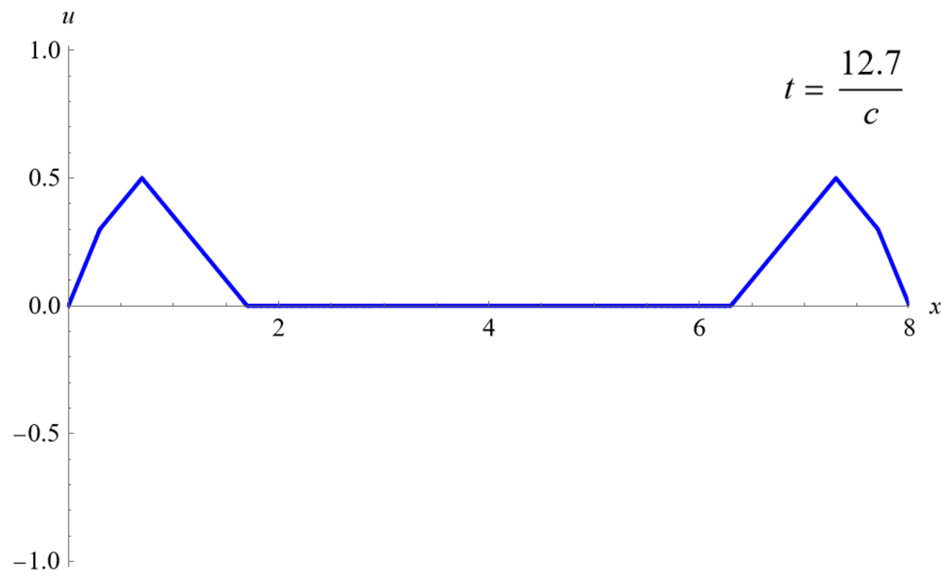
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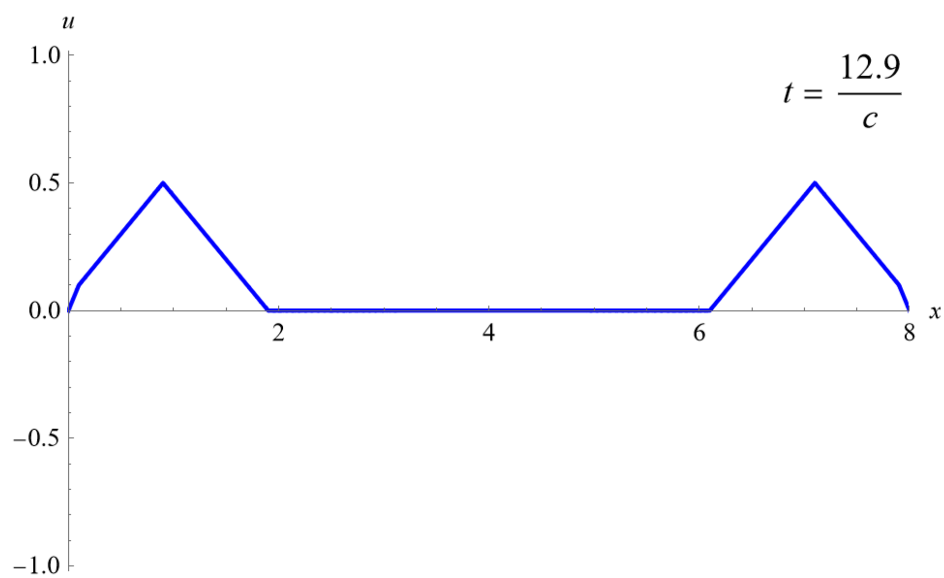
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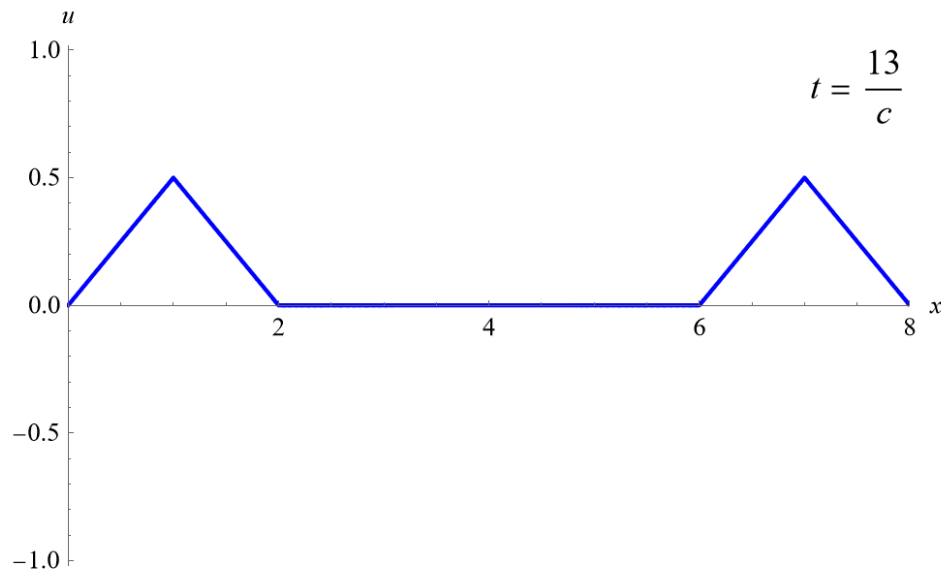
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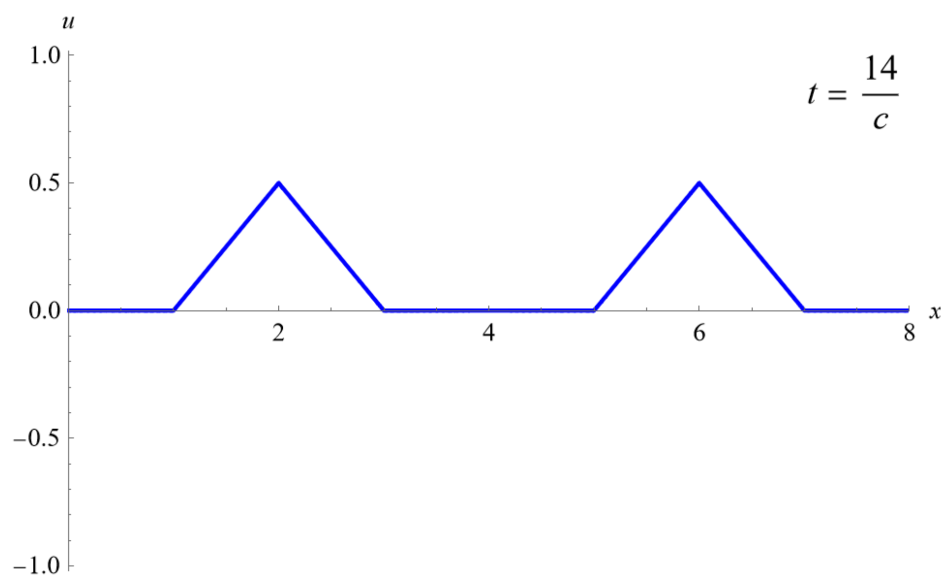
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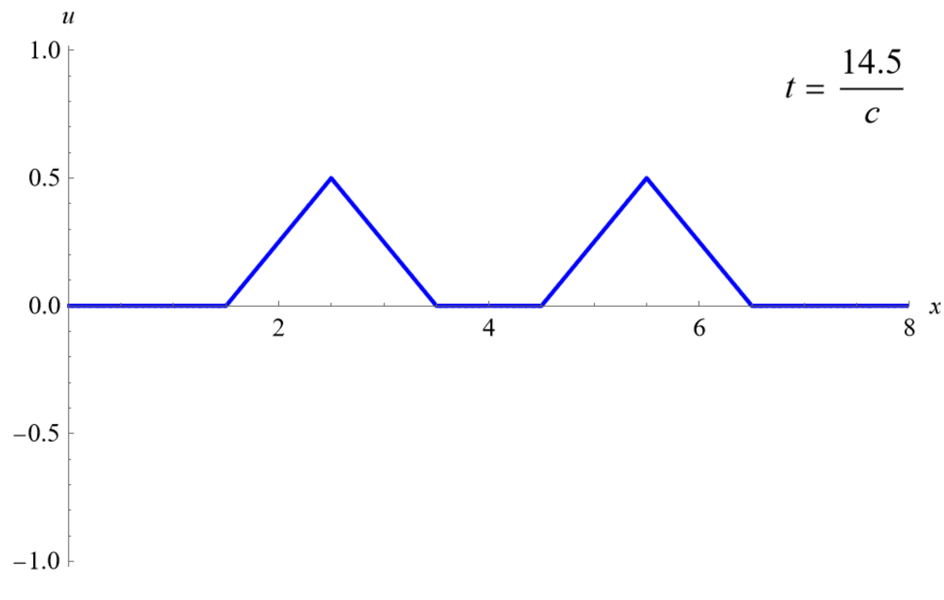
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x
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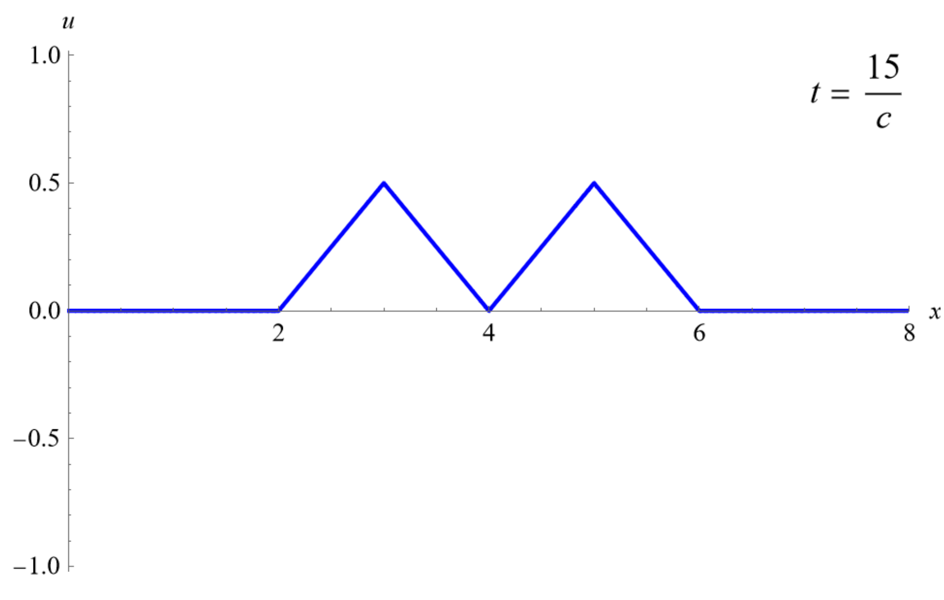
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8
x
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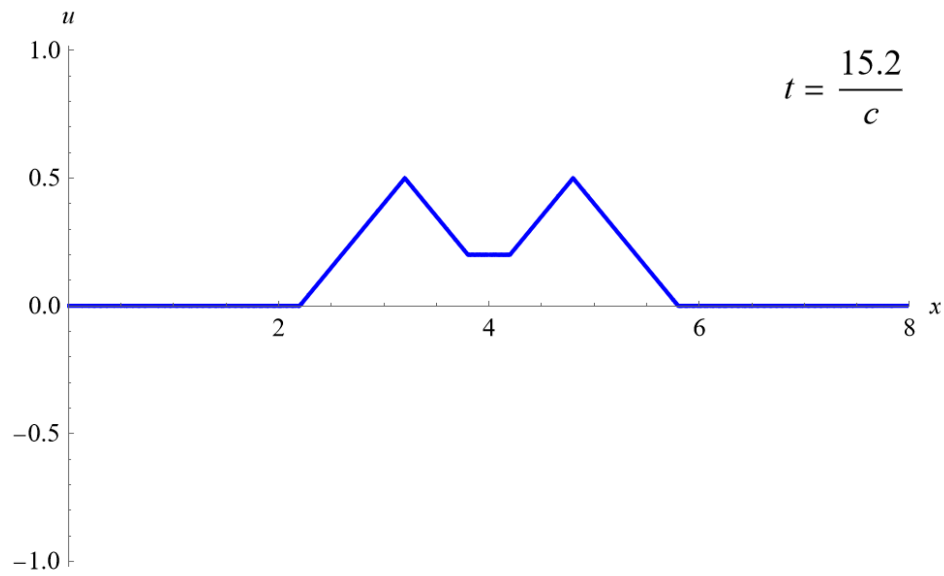
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6
8
x
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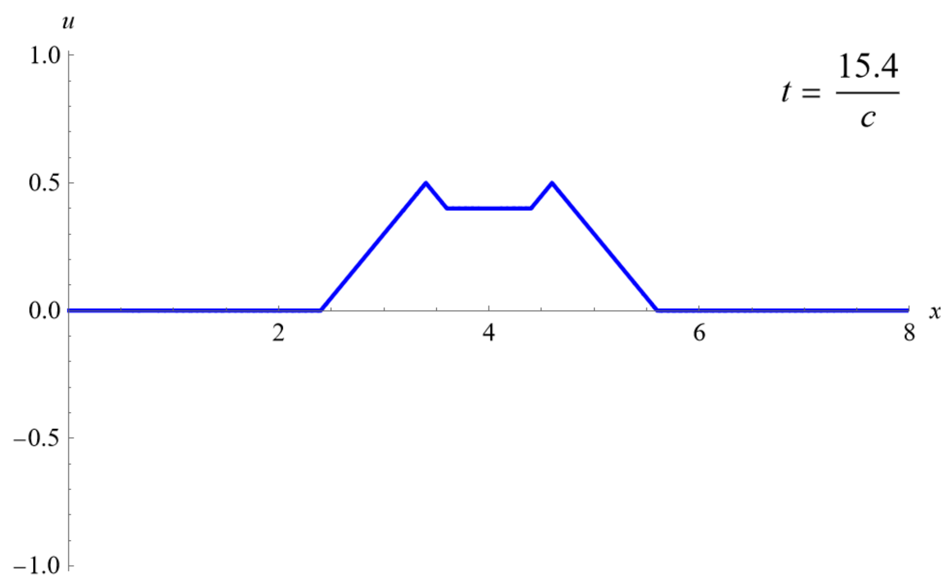
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6
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x
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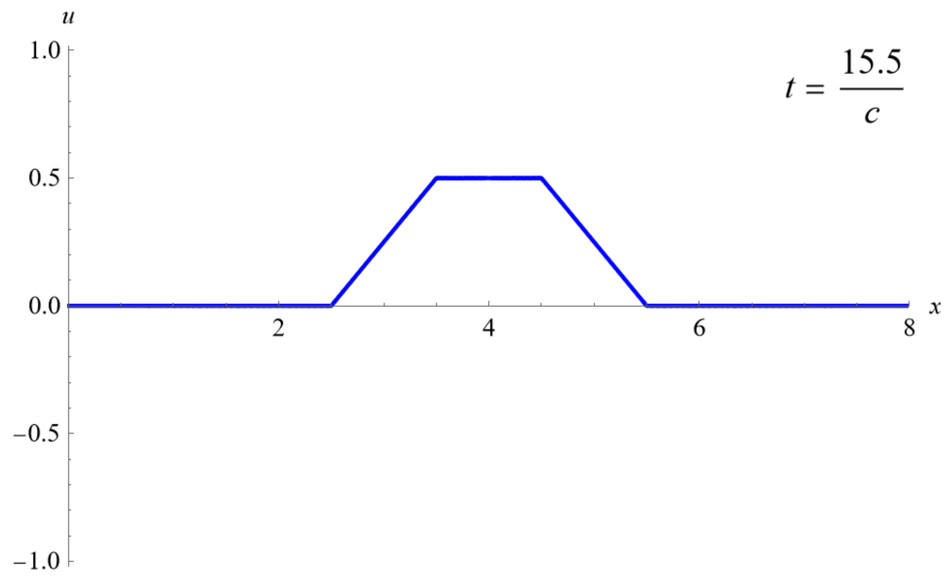
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x
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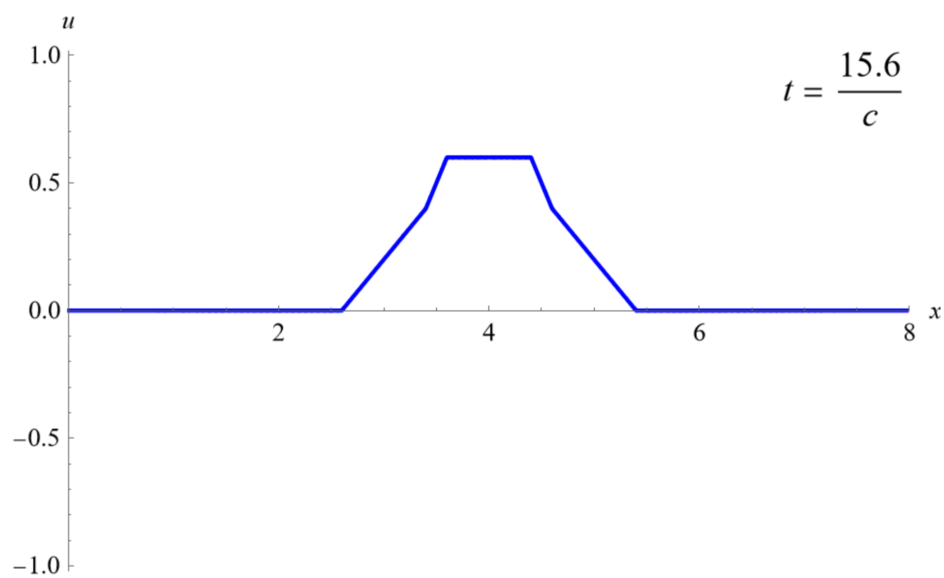
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4
6
8
x
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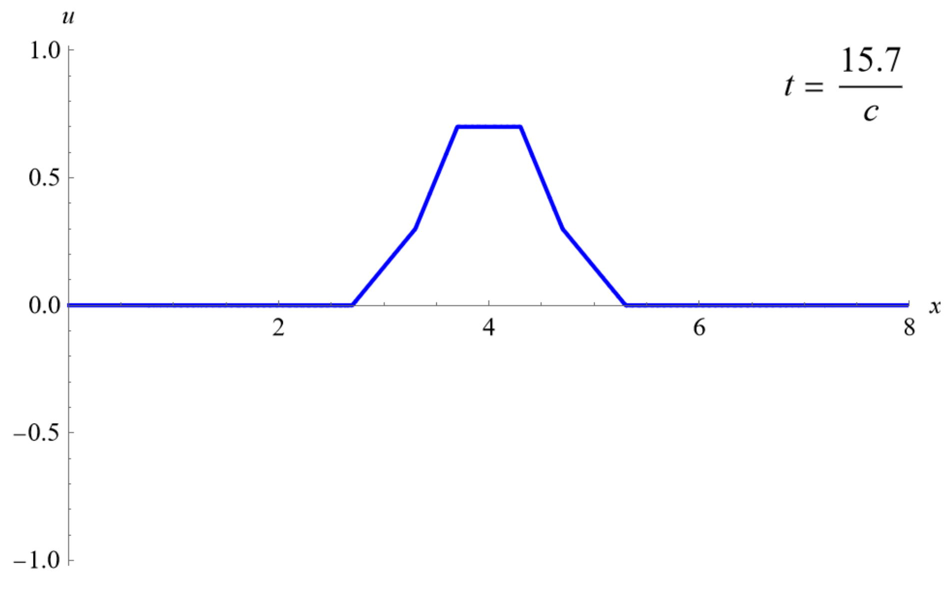
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4
6
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x
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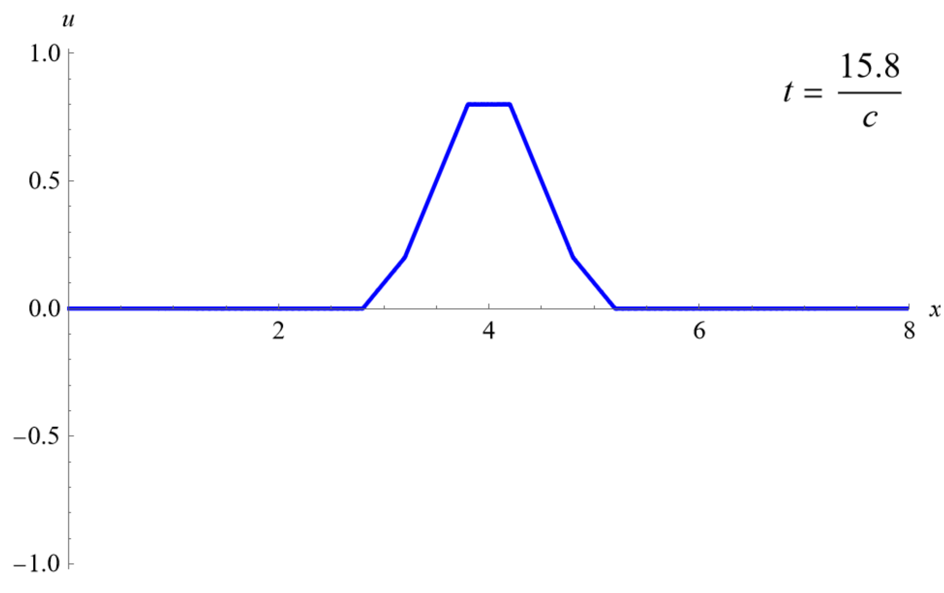
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6
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x
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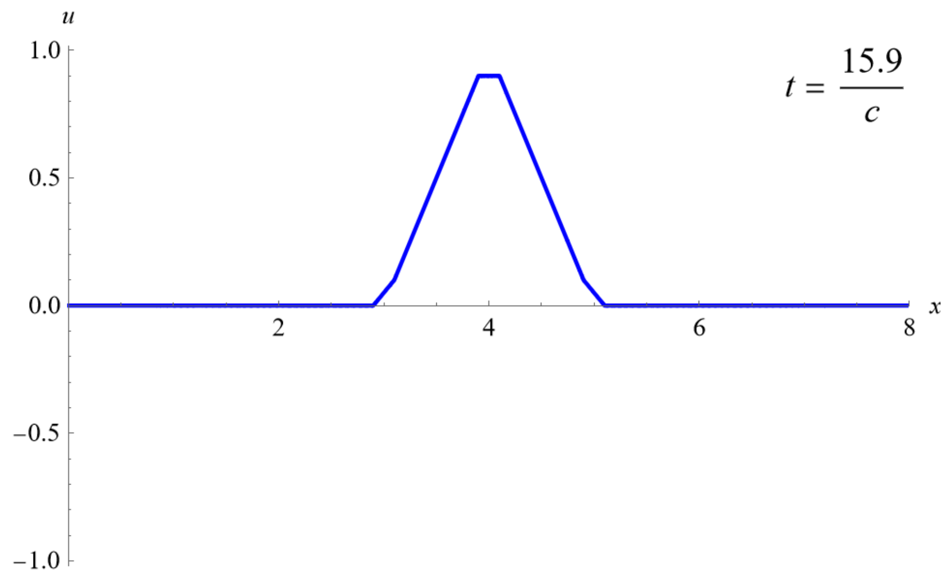

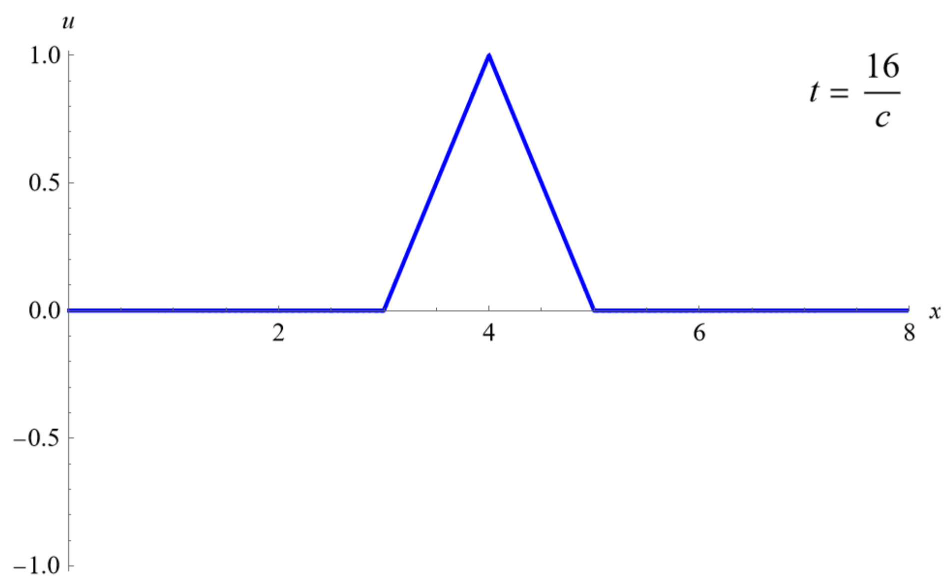

As the triangles come together, they interfere constructively, producing a triangle with a height of 1.